

THE CHARACTERIZATION OF THETA-DISTINGUISHED REPRESENTATIONS OF GL_n

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ABSTRACT. Let θ and θ' be a pair of exceptional representations in the sense of Kazhdan and Patterson [KP84], of a metaplectic double cover of GL_n . The tensor $\theta \otimes \theta'$ is a (very large) representation of GL_n . We characterize its irreducible generic quotients. In the square-integrable case, these are precisely the representations whose symmetric square L -function has a pole at $s = 0$. Our proof of this case involves a new globalization result. In the general case these are the representations induced from distinguished data or pairs of representations and their contragredients. The combinatorial analysis is based on a complete determination of the twisted Jacquet modules of θ . As a corollary, θ is shown to admit a new “metaplectic Shalika model”.

1. INTRODUCTION

Let F be a local non-Archimedean field of characteristic zero. Let θ and θ' be a pair of exceptional representations in the sense of Kazhdan and Patterson [KP84], these are representations of a metaplectic double cover $\widetilde{\mathrm{GL}}_n = \widetilde{\mathrm{GL}}_n(F)$ of $\mathrm{GL}_n = \mathrm{GL}_n(F)$. The GL_n -module $\theta \otimes \theta'$ has appeared, in both local and global incarnations, in several studies, most notably in the work of Bump and Ginzburg [BG92] on the global symmetric square L -function. Locally it has been studied by Savin [Sav92] and Kable [Kab01, Kab02], who considered multiplicity-one properties and spherical quotients. In a more general context, exceptional or minimal representations have played an important role in the theta correspondence, the descent method and Rankin-Selberg integrals [GRS03, Gin06, GJS11].

Our main goal in this work is to characterize the generic quotients of $\theta \otimes \theta'$. An admissible representation τ of GL_n is called distinguished if

$$\mathrm{Hom}_{\mathrm{GL}_n}(\theta \otimes \theta', \tau^\vee) \neq 0,$$

where τ^\vee is the representation contragradient to τ . Here is our main result.

Theorem A. *Let τ be an irreducible generic representation of GL_n .*

- (1) *If τ is essentially square-integrable, then it is distinguished if and only if it is unitary and the symmetric square L -function $L(s, \tau, \mathrm{Sym}^2)$ has a pole at $s = 0$.*
- (2) *In general write τ as a parabolically induced representation $\tau = \tau_1 \times \dots \times \tau_m$, where each τ_i is essentially square-integrable. Then τ is distinguished if and only if there is $0 \leq m_0 \leq \lfloor m/2 \rfloor$ such that, perhaps after permuting the indices of the inducing data, $\tau_{2i} = \tau_{2i-1}^\vee$ for $1 \leq i \leq m_0$ and τ_i is distinguished for $2m_0 + 1 \leq i \leq m$.*

The supercuspidal case has already been proved in [Kap14c]. The square-integrable case is handled in Theorem 4.12. It relies on a new globalization result (see below). In this case there is one assumption, caused by the globalization method: when τ is square-integrable with a nontrivial central character and $L(s, \tau, \mathrm{Sym}^2)$ has a pole at $s = 0$, the proof that τ is distinguished relies on the existence of the local functorial lift of Cogdell et al. [CKPSS04]

for quasi-split special even orthogonal groups (the global lift was described in [CPSS11]). See Remark 4.11 for more details.

The fact that parabolic induction (irreducible or not) of distinguished representations is also distinguished, i.e., an upper heredity result, was established in [Kap14b]. In Theorem 4.4 we show that for an irreducible τ (generic or not), $\tau \times \tau^\vee$ is always distinguished. The structure of generic distinguished representations in general is given in Theorem 4.16.

Now consider, for example, the representation $\nu^{1/2}\rho \times \nu^{-1/2}\rho$, where $\nu = |\det|$ and ρ is an irreducible unitary self-dual supercuspidal representation of GL_n . It is of length 2, has a unique irreducible quotient $\mathrm{Lang}(\nu^{1/2}\rho \times \nu^{-1/2}\rho)$ - the Langlands quotient, and a unique irreducible subrepresentation Δ , which is square-integrable ([Zel80] 9.1). Theorem 4.4 implies that $\nu^{1/2}\rho \times \nu^{-1/2}\rho$ is distinguished. Then Theorem A shows that, depending on the pole of $L(s, \Delta, \mathrm{Sym}^2)$ at $s = 0$, Δ is distinguished or not. But in the absence of a pole, we deduce that $\mathrm{Lang}(\nu^{1/2}\rho \times \nu^{-1/2}\rho)$ is distinguished. This is an example of a non-generic distinguished representation.

Another immediate corollary of Theorem A is that an irreducible generic distinguished representation must be self-dual.

Savin [Sav92] considered the spherical quotients of $\theta \otimes \theta'$. He conjectured that a spherical representation with a trivial central character is distinguished, if and only if it is the lift of a representation of a prescribed classical group. The case of $n = 3$ was established in [Sav92]. Kable [Kab02] proved that such lifts are distinguished, for a general n , using analytic methods. The other direction was settled in [Kap14b], by extending the ideas of [Sav92].

In more detail, let $B_n = T_n \ltimes N_n$ be the Borel subgroup of GL_n , where N_n is the subgroup of upper triangular unipotent matrices, and $Q_{n-k,k} = M_{n-k,k} \ltimes U_{n-k,k}$ be the standard maximal parabolic subgroup with $M_{n-k,k} \cong \mathrm{GL}_{n-k} \times \mathrm{GL}_k$. In [Kap14b] we described a filtration of $(\theta \otimes \theta')_{N_n}$ as a T_n -module, using the theory of derivatives of Bernstein and Zelevinsky [BZ76, BZ77]. The techniques of [Kap14b] break down in the general setting, where we have a Jacquet module of $\theta \otimes \theta'$ with respect to a unipotent radical of an arbitrary standard parabolic subgroup. The theory of derivatives is no longer applicable, the main barrier being that the action of $M_{n-k,k}$ on the set of nontrivial characters of $U_{n-k,k}$ is not transitive.

In fact, the arguments here can also be used to deduce the structure of distinguished spherical representations, but the claims in [Kap14b] are actually stronger: the appearance of the inducing character as a subquotient of $(\theta \otimes \theta')_{N_n}$ determines its combinatorial structure completely.

In this work we take a different approach, and rely on the computation of all the twisted Jacquet modules of θ , corresponding to maximal parabolic subgroups. Let $0 < k < n$ and fix an additive character ψ of F . The action of $M_{n-k,k}$ on the characters of $U_{n-k,k}$ has $\min(n-k, k) + 1$ orbits, we fix representatives ψ_j , where $0 \leq j \leq \min(n-k, k)$. The character ψ_0 is the trivial character, and when $n = 2k$, ψ_k is the generic character in the sense that its stabilizer \mathbb{GL}_k in $M_{k,k}$ is reductive. The group \mathbb{GL}_k is simply the diagonal embedding of GL_k in $M_{k,k}$. Restriction of the cover of GL_{2k} to \mathbb{GL}_k is a trivial double cover (\mathbb{GL}_k splits under this cover).

Theorem B. *For any $0 \leq j \leq \min(n-k, k)$, the Jacquet module of θ with respect to $U_{n-k,k}$ and ψ_j is isomorphic to $(\theta_{n-k-j} \widetilde{\otimes} \theta_{k-j}) \otimes \gamma_{\psi, (-1)^{j-1}} \psi$. Here θ_m is an exceptional representation*

of $\widetilde{\mathrm{GL}}_m$, $\widetilde{\otimes}$ is the metaplectic tensor of Kable [Kab01], $\gamma_{\psi,(-1)^{j-1}\psi}$ is a one-dimensional representation of $\mathrm{GL}_j \ltimes U_{j,j}$, and $\gamma_{\psi,(-1)^{j-1}}$ is a certain Weil factor regarded as a character of $\widetilde{\mathrm{GL}}_j$.

The main ingredient in the proof is the case of $n = 2k = 2j$. When $n = 2$, ψ_1 is the usual Whittaker character of N_2 and this result was proved by Gelbart and Piatetski-Shapiro [GPS80] (Theorem 2.2). The general case is proved in Theorem 3.1 using induction and the local “exchange of roots” of Ginzburg, Rallis and Soudry [GRS99]. For the global analog, one may utilize the global “exchanging roots” ([Gin90, GRS01, Sou05, GRS11]). The result for an arbitrary maximal parabolic is given in Theorem 3.3.

Kable [Kab01] computed the nontwisted Jacquet modules, i.e., with the trivial character ψ_0 , and expressed them as metaplectic tensors. This already had several applications [Kab01, Kab02, Kap14b].

The particular case of $n = 2k$ and the generic character ψ_k implies a notion of a “metaplectic Shalika model” (ψ_k takes the form of the usual Shalika character). In this case, Theorem B implies the existence of a unique (up to a scalar) nontrivial linear functional on the space of θ , which is ψ_k -equivariant on the left under $U_{k,k}$. In the non-metaplectic setting, for the existence of a Shalika functional, the stabilizer GL_k must then act trivially. The correspondence between representations of GL_k and $\widetilde{\mathrm{GL}}_k$ (recall that this cover is trivial) takes 1 to $1 \otimes \gamma_\psi$, which is essentially the representation $\gamma_{\psi,(-1)^{j-1}}$ we obtained. In turn, we deduce a metaplectic Shalika model for θ , which we use for the proof of Theorem 4.4. For more details see Remark 3.2. We mention that this model may have applications similar to those of [FJ93, JR96].

Let GSpin_{2n+1} be the split odd general spin group of rank $n + 1$. A result similar to Theorem B, for exceptional representations Θ of GSpin_{2n+1} (defined in [Kap14a] following the exposition in [BFG03] for SO_{2n+1}), was proved in [Kap14c] and used in a study of $\Theta \otimes \Theta'$. The definition of distinguished representations of GSpin_{2n+1} is similar to that of GL_n , an irreducible representation is distinguished if its contragradient is a quotient of $\Theta \otimes \Theta'$. The results of [Kap14c] allow us to alternate between quotients of $\theta \otimes \theta'$ and $\Theta \otimes \Theta'$, using parabolic induction.

One essential difference between these settings, is that in contrast with $\theta \otimes \theta'$, the space $\Theta \otimes \Theta'$ does not afford a Whittaker functional ([Kap14c] Theorem 1). This interplay will be used as an ingredient in a forthcoming work on the conjecture of Lapid and Mao on Whittaker-Fourier coefficients, for even orthogonal groups ([LM13]). One can also use this to deduce certain irreducibility results (see Remark 4.15). In more detail, Proposition 4.1 of [Kap14c] showed that for a tempered distinguished τ , a representation parabolically induced from $\nu^{1/2}\tau \otimes 1$ to GSpin_{2n+1} will also be distinguished, but then it must be reducible, otherwise it is a generic quotient of $\Theta \otimes \Theta'$.

A result similar to a global formulation of Theorem B was used to compute a global co-period integral, involving the integration of the residue of an Eisenstein series, against a pair of automorphic forms in the spaces of the small representation of SO_{2n+1} or GSpin_{2n+1} [Kap, Kap14a].

To prove that a square-integrable representation τ such that $L(s, \tau, \mathrm{Sym}^2)$ has a pole at $s = 0$, is distinguished, we apply the following globalization lemma. As mentioned above, in one case its proof relies on an assumption, explicitly given in Remark 4.11.

Theorem C. *Let π be a square-integrable representation of GL_n . Assume that $L(s, \pi, R)$ has a pole at $s = 0$, for $R = \mathrm{Sym}^2$ or \wedge^2 . Then there exist a number field with a ring of adèles \mathbb{A} and a global cuspidal representation Π of $\mathrm{GL}_n(\mathbb{A})$, such that for a sufficiently large finite set of places S , $L^S(s, \Pi, R)$ has a pole at $s = 1$, and at some place v , $\Pi_v = \pi$.*

See Lemma 4.9. This result is expected to have other applications.

There is a minor “historical gap” in the theory of exceptional representations, regarding their Whittaker models. Let F be any local non-Archimedean field of characteristic different from 2. Kazhdan and Patterson [KP84] proved that for $n \geq 3$, if $|2| \neq 1$ in F , the exceptional representations do not have Whittaker models. For $n = 3$, Flicker, Kazhdan and Savin [FKS90] (Lemma 6) used global methods to extend this result to the case $|2| = 1$. The remaining case ($|2| = 1$ and $n > 3$) has been expected, but not proved. We complete the proof in Theorem 2.6. This immediately validates several results from [BG92, Kab01, Kab02, Kap14b] when the field is dyadic, including the local functional equation of [BG92] (Section 5) and the aforementioned conjecture of Savin (see [Kab02] p. 1602), which now hold in general.

Exceptional representations for $\widetilde{\mathrm{GL}}_n$ were introduced and studied by Kazhdan and Patterson [KP84]. They are related to a broader class of small, or minimal, representations. The first example was probably the Weil representation of $\widetilde{\mathrm{Sp}}_n$. These representations enjoy the vanishing of a large class of Fourier coefficients, which makes them valuable for applications involving lifts and Rankin-Selberg integrals [GRS03, Gin06, GJS11]. They have had a profound role in the theory of representations and appeared in numerous studies, including [Vog81, Kaz90, KS90, Sav93, BK94, Sav94, GRS97, BFG00, GRS01, KPW02, BFG03, JS03, GS05, Sou06, LS08, GRS11].

To date, the significant application of the exceptional representations of Kazhdan and Patterson [KP84], is the construction of a Rankin-Selberg integral representation for the symmetric square L -function, by Bump and Ginzburg [BG92]. Takeda [Tak14] extended the results of [BG92], to some extent, to the twisted symmetric square L -function.

The term “distinguished” has been used in various contexts. Let ξ be a representation of a group G and let η be a character of a subgroup $H < G$. The representation ξ is called (H, η) -distinguished, if $\mathrm{Hom}_H(\xi, \eta) \neq 0$. There are numerous studies on local and global distinguished representations, including [Jac91, JR92, FJ93, Off06, OS07, OS08, Off09, Jac10, Mat10a, Mat10b, Mat11, FLO12, Mat].

Let F_0 be a quadratic extension of F . Matringe [Mat10a, Mat10b, Mat11] studied $(\mathrm{GL}_n(F), \eta)$ -distinguished representations of $\mathrm{GL}_n(F_0)$. He proved, using different (but in some sense related) techniques, a combinatorial characterization similar to Theorem A. He also proved ([Mat10b]) that an irreducible generic representation ξ is distinguished, if and only if its Rankin-Selberg Asai L -function has an exceptional pole at 0. He then showed that $L(s, \xi, \mathrm{Asai}) = L(s, \rho(\xi), \mathrm{Asai})$, where $\rho(\xi)$ is the Langlands parameter associated with ξ ([Mat09]).

Feigon, Lapid and Offen [FLO12] studied (local and global) representations distinguished by unitary groups.

The rest of this work is organized as follows. Section 2 contains the preliminaries, including a brief review of the metaplectic tensor of Kable [Kab01] and exceptional representations. The Jacquet modules are computed in Section 3. Our results on distinguished representations occupy Section 4.

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2. PRELIMINARIES

2.1. The groups. Let F be a local non-Archimedean field of characteristic different from 2. Let $(,)_2$ be the Hilbert symbol of order 2 of F and put $\mu_2 = \{-1, 1\}$. For a group G , C_G denotes its center. If $x, y \in G$ and $Y < G$, $xy = xyx^{-1}$, ${}^xY = \{xy : y \in Y\}$. Also if $d \in \mathbb{Z}$, $Y^d = \{y^d : y \in Y\}$ and in particular $F^{*d} = (F^*)^d$. The set of $m \times k$ matrices over F is denoted $F^{m \times k}$.

Note that our results up to and including Corollary 4.7 apply to any local p -adic field of characteristic different from 2. Then we add the assumption that the characteristic of F is 0 (see also Remark 4.18). Henceforth we omit references to the field, e.g., $\mathrm{GL}_n = \mathrm{GL}_n(F)$.

Fix the Borel subgroup of upper triangular invertible matrices $B_n = T_n \ltimes N_n$, where T_n is the diagonal torus. A partition of n into m parts is an m -tuple $\alpha = (\alpha_1, \dots, \alpha_m)$ of nonnegative integers, whose sum is n (partitions will be regarded as ordered partitions). For a partition α , let $Q_\alpha = M_\alpha \ltimes U_\alpha$ denote the corresponding standard parabolic subgroup; its Levi part M_α is isomorphic to $\mathrm{GL}_{\alpha_1} \times \dots \times \mathrm{GL}_{\alpha_m}$. In particular the maximal parabolic subgroups are the subgroups $Q_{n-k,k}$ for $0 \leq k \leq n$; $Q_n = Q_{0,n} = Q_{n,0} = \mathrm{GL}_n$. The group GL_k is regarded as a subgroup of GL_n via its natural embedding in $M_{k,n-k}$. For any parabolic subgroup $Q < \mathrm{GL}_n$, let δ_Q denote its modulus character.

Let $\widetilde{\mathrm{GL}}_n$ be the metaplectic double cover of GL_n , as constructed by Kazhdan and Patterson [KP84] (with their c parameter equal to 0). They defined their cover by restriction from the cover of SL_{n+1} of Matsumoto [Mat69]. We use the block-compatible cocycle $\sigma = \sigma_n$ of Banks, Levi and Sepanski [BLS99], which coincides with the cocycle of Kubota [Kub67] for $n = 2$. The block-compatibility property reads, for $a, a' \in \mathrm{GL}_{n-k}$ and $b, b' \in \mathrm{GL}_k$,

$$(2.1) \quad \sigma(\mathrm{diag}(a, b), \mathrm{diag}(a', b')) = \sigma_{n-k}(a, a') \sigma_k(b, b') (\det a, \det b')_2.$$

Let $\mathfrak{s} : \mathrm{GL}_n \rightarrow \widetilde{\mathrm{GL}}_n$ be the section of [BLS99] and $p : \widetilde{\mathrm{GL}}_n \rightarrow \mathrm{GL}_n$ be the natural projection. For any subset $X \subset \mathrm{GL}_n(F)$, denote $\widetilde{X} = p^{-1}(X)$. We pull back the determinant to a non-genuine character of $\widetilde{\mathrm{GL}}_n$, also denoted \det , and using this any character of F^* can be pulled back to a character of $\widetilde{\mathrm{GL}}_n$. Let e be 1 if n is odd, otherwise $e = 2$. Then $C_{\widetilde{\mathrm{GL}}_n} = p^{-1}(C_{\mathrm{GL}_n}^e)$.

2.2. Representations. Let G be an l -group ([BZ76] 1.1). Representations of G will be complex and smooth. Let $\mathrm{Alg} G$ denote the category of these representations and $\mathrm{Alg}_{\mathrm{irr}} G \subset \mathrm{Alg} G$ the subcategory of irreducible representations. For $\pi \in \mathrm{Alg} G$, π^\vee is the representation contragradient to π . If π admits a central character, it will be denoted ω_π . If $H < G$ and $g \in G$, ${}^g\pi$ is the representation of gH on the space of π given by ${}^g\pi(x) = \pi(g^{-1}x)$. We say that π is glued from representations π_1, \dots, π_k , if π has a filtration, whose quotients (which may be isomorphic or zero) are, after a permutation, π_1, \dots, π_k .

Assume that \widetilde{G} is a given central extension of G by μ_2 and $\varphi : G \rightarrow \widetilde{G}$ is a section. If π and π' are genuine representations of \widetilde{G} , $\pi \otimes \pi'$ (outer tensor product) is a representation of G via $g \mapsto \pi(\varphi(g)) \otimes \pi'(\varphi(g))$. The definition is independent of the choice of φ and the actual section will be omitted.

Let X be an l -space ([BZ76] 1.1). The space of Schwartz-Bruhat functions on X is denoted $\mathcal{S}(X)$. If G acts on X , it also acts on $\mathcal{S}(X)$.

Regular induction is denoted Ind and ind is the compact induction. Induction is not normalized. In GL_n , if $\alpha = (\alpha_1, \dots, \alpha_m)$ is a partition of n and $\tau_1 \otimes \dots \otimes \tau_m \in \text{Alg } M_\alpha$, i.e., $\tau_i \in \text{Alg } \text{GL}_{\alpha_i}$, then $\tau_1 \times \dots \times \tau_m = \text{Ind}_{Q_\alpha}^{\text{GL}_n}(\delta_{Q_\alpha}^{1/2} \tau_1 \otimes \dots \otimes \tau_m)$.

Let $\text{Alg}_{\text{sqr}} \text{GL}_n \subset \text{Alg}_{\text{esqr}} \text{GL}_n \subset \text{Alg}_{\text{irr}} \text{GL}_n$ be the subcategories of square-integrable and essentially square-integrable representations. We briefly recall the characterization of essentially square-integrable representations of Zelevinsky [Zel80] (Section 9). Let $\text{Alg}_{\text{cusp}} \text{GL}_n \subset \text{Alg}_{\text{esqr}} \text{GL}_n$ denote the subcategory of supercuspidal representations (not necessarily unitary). For brevity, put $\nu = |\det|$. Let \mathcal{C} be the set of equivalence classes of $\text{Alg}_{\text{cusp}} \text{GL}_n$ for all $n \geq 0$. A segment in \mathcal{C} is a subset $[\rho, \nu^{l-1}\rho] = \{\rho, \nu\rho, \dots, \nu^{l-1}\rho\}$, where $\rho \in \text{Alg}_{\text{cusp}} \text{GL}_k$ and $l \geq 1$ is an integer. The representation $\rho \times \nu\rho \times \dots \times \nu^{l-1}\rho$ has a unique irreducible quotient denoted $\langle [\rho, \nu^{l-1}\rho] \rangle^t$. Then $\tau \in \text{Alg}_{\text{esqr}} \text{GL}_n$ if and only if it is isomorphic to some representation $\langle [\rho, \nu^{l-1}\rho] \rangle^t$. In this case $\omega_\tau = \nu^{(l-1)/2} \omega_\rho$. Also note that $(\langle [\rho, \nu^{l-1}\rho] \rangle^t)^\vee = \langle [\nu^{1-l}\rho^\vee, \rho^\vee] \rangle^t$.

Zelevinsky [Zel80] also described the reducibility of products of essentially square-integrable representations. Two segments $[\rho, \nu^{l-1}\rho]$ and $[\rho', \nu'^{l'-1}\rho']$ are called linked, if neither of them is a subset of the other, but their union is a segment. If they are linked, in particular $\rho = \nu^a \rho'$ for some integer a . The representation $\langle [\rho_1, \nu^{l_1-1}\rho_1] \rangle^t \times \dots \times \langle [\rho_m, \nu^{l_m-1}\rho_m] \rangle^t$ is irreducible if and only if no pair of segments are linked.

Let ψ be a nontrivial additive character of F . Denote the normalized Weil factor by γ_ψ ([Wei64] Section 14), $\gamma_\psi(\cdot)^4 = 1$. For $a \in F^*$, let $\gamma_{\psi,a}$ be the Weil factor corresponding to the character $x \mapsto \psi(ax)$. Recall the following formulas (see the appendix of [Rao93]):

$$(2.2) \quad \gamma_\psi(xy) = \gamma_\psi(x)\gamma_\psi(y)(x, y)_2, \quad \gamma_\psi(x^2) = 1, \quad \gamma_{\psi,a}(x) = (a, x)_2 \gamma_\psi(x).$$

We also denote by ψ the generic character of N_n given by $\psi(u) = \psi(\sum_{i=1}^{n-1} u_{i,i+1})$. By restriction, it is a character of any unipotent subgroup $U < N_n$. When we define a character of such a subgroup, we usually use the notation ψ , if it coincides with this restriction.

Let $\tau \in \text{Alg}_{\text{irr}} \text{GL}_n$. A (ψ) -Whittaker functional on τ is a functional λ such that $\lambda(u\varphi) = \psi(u)\lambda(\varphi)$ for any $u \in N_n$ and φ in the space of τ . We say that τ is generic, if it admits a nontrivial Whittaker functional.

If $n = 2k$ we can also consider a Shalika functional, the equivariance property now reads $\lambda((\begin{smallmatrix} c & \\ & v \end{smallmatrix})\varphi) = \psi(\text{tr}(v))\lambda(\varphi)$ for any $c \in \text{GL}_k$ and $v \in F^{k \times k}$. If $\lambda \neq 0$, it yields an embedding $\tau \subset \text{Ind}_{\text{GL}_k U}^{\text{GL}_{2k}}(\psi)$, where $\text{GL}_k = \{\text{diag}(c, c) : c \in \text{GL}_k\}$, called a Shalika model of τ . If it exists, uniqueness of this model was proved by Jacquet and Rallis [JR96] in the non-Archimedean case and by Ash and Ginburg [AG94] over Archimedean fields.

2.3. Jacquet modules. Let $\pi \in \text{Alg } G$. Let $U < G$ be a closed subgroup, exhausted by its compact subgroups (here U will be a unipotent subgroup of GL_n) and ψ be a character of U . Assume $M < G$ is the normalizer of U and stabilizer of ψ . The Jacquet module of π with respect to U and ψ is denoted $\pi_{U,\psi}$. The action is not normalized. We have the following exact sequence in $\text{Alg } M$,

$$0 \rightarrow \pi(U, \psi) \rightarrow \pi \rightarrow \pi_{U,\psi} \rightarrow 0.$$

The representation $\pi(U, \psi)$ can be characterized by the Jacquet-Langlands lemma:

Lemma 2.1. (see e.g. [BZ76] 2.33) *a vector v in the space of π belongs to $\pi(U, \psi)$ if and only if*

$$\int_U \pi(u) v \psi^{-1}(u) du = 0,$$

for some compact subgroup $\mathcal{U} < U$.

When $\psi = 1$, we simply write $\pi(U)$ and π_U . The following consequence of this lemma will be used.

Lemma 2.2. *Let Q be an l -group and $U < Q$ be a closed subgroup, exhausted by its compact subgroups, which is normal in Q . The group Q acts on the set of characters ψ of U by $\psi^g(u) = \psi(g^{-1}u)$, denote an orbit by $\mathcal{O}(\psi)$. Assume ψ and ψ' are two characters of U and let $L, L' < Q$ be their stabilizers. Assume $L, L', LU, L'U$ are closed subgroups of Q . Then for $\pi \in \text{Alg } LU$ and $\pi' \in \text{Alg } L'U$,*

$$(\text{ind}_{LU}^Q \pi_{U,\psi} \otimes \text{ind}_{L'U}^Q \pi'_{U,\psi'})_U = \begin{cases} \text{ind}_{LU}^Q (\pi_{U,\psi} \otimes \pi'_{U,\psi^{-1}}) & \psi' = \psi^{-1}, \\ 0 & \psi' \notin \mathcal{O}(\psi^{-1}). \end{cases}$$

Proof of Lemma 2.2. Let f belong to the space of $\text{ind}_{LU}^Q \pi_{U,\psi}$. Choose a small compact open neighborhood of the identity $\mathcal{V} < Q$ and a finite set $\Omega \subset Q$, such that f is fixed by \mathcal{V} and the support of f is contained in $LU\Omega\mathcal{V}$. Similarly let f' belong to the space of $\text{ind}_{L'U}^Q \pi'_{U,\psi'}$ and choose \mathcal{V}' and Ω' , analogously. For any compact $\mathcal{U} < U$ and $g, g' \in Q$,

$$(2.3) \quad \int_{\mathcal{U}} u \cdot (f \otimes f')(g, g') \, du = f(g) \otimes f'(g') \int_{\mathcal{U}} \psi(gu) \psi'(g'u) \, du.$$

This clearly vanishes unless $g = luxv$, where $l \in L$, $u \in U$, $x \in \Omega$ and $v \in \mathcal{V}$, and $g' = l'u'x'v'$ (with a similar notation). In this case the integral equals

$$(2.4) \quad \psi(u) \pi_{U,\psi}(l) f(x) \otimes \psi'(u) \pi'_{U,\psi'}(l') f'(x') \int_{\mathcal{U}} \psi(xvu) \psi'(x'v'u) \, du.$$

Assume $\psi' \notin \mathcal{O}(\psi^{-1})$. Then for any $y, y' \in Q$, $u \mapsto \psi(yu) \psi'(y'u)$ is a nontrivial character of U . Since x and x' vary in finite sets, while v and v' vary in compact neighborhoods of the identity, one can choose a large enough compact \mathcal{U} such that the integral vanishes for all data x, v, x', v' . Thus Lemma 2.1 implies that $f \otimes f'$ vanishes in the Jacquet module.

Now assume $\psi' = \psi^{-1}$. Consider the mapping

$$f \otimes f' \mapsto (f \otimes f')(g) = f(g) \otimes f'(g).$$

It is clearly onto $\text{ind}_{LU}^Q (\pi_{U,\psi} \otimes \pi'_{U,\psi^{-1}})$. If $f \otimes f' = 0$ in the Jacquet module, by lemma 2.1 the left-hand side of (2.3) vanishes for all $g = g' \in Q$, but then the du -integral on the right-hand side becomes a nonzero constant. This implies $(f \otimes f')(g) = f(g) \otimes f'(g) = 0$.

In the other direction assume $f(g) \otimes f'(g) = 0$ for all $g \in Q$. We must show that (2.3) vanishes for all $g, g' \in Q$. It clearly does if $g = g'$. Take $g \neq g'$. Using the notation above write $g = luxv$ and $g' = l'u'x'v'$ and consider the right-hand side of (2.3).

If the character $u \mapsto \psi(gu) \psi^{-1}(g'u)$ is trivial on U , $g'g^{-1} = l''u'' \in LU$, but then

$$f(g) \otimes f'(g') = f(g) \otimes \psi^{-1}(u'') \pi'_{U,\psi^{-1}}(l'') f'(g').$$

By our assumption either $f(g) = 0$ or $f'(g) = 0$ and in the latter case, $\pi'_{U,\psi^{-1}}(l'') f'(g) = 0$ for any $l'' \in L$. Thus $f(g) \otimes f'(g') = 0$.

For a pair g, g' such that this character is a nontrivial character, as above using the fact that x, x' vary in finite sets and v, v' belong to compact subgroups, one can choose a large enough \mathcal{U} so that this integral vanishes.

Finally to pass from pure tensors to the general case, note that a general element of $\text{ind}_{LU}^Q \pi_{U,\psi} \otimes \text{ind}_{L'U}^Q \pi'_{U,\psi^{-1}}$ can be written as a finite sum $\sum_i f_i \otimes f'_i$ with the following property:

for any subset of indices i_1, \dots, i_r such that the supports of $f_{i_j} \otimes f'_{i_j}$ coincide for $1 \leq j \leq r$, for each g, g' in the support of $f_{i_1} \otimes f'_{i_1}$, the vectors $(f_{i_1} \otimes f'_{i_1})(g, g'), \dots, (f_{i_r} \otimes f'_{i_r})(g, g')$ are linearly independent in $\pi_{U, \psi} \otimes \pi'_{U, \psi^{-1}}$. Then $(\sum_i f_i \otimes f'_i)(g, g') = 0$ if and only if for each i , $(f_i \otimes f'_i)(g, g') = 0$. \square

2.4. Metaplectic tensor. Let α be a partition of n . Irreducible representations of M_α can be described using the tensor product. The situation in \widetilde{M}_α is more complicated, because the direct factors of M_α do not commute in the cover, then the tensor construction breaks down. An alternative definition of a metaplectic tensor was developed by Kable [Kab01], we briefly describe his construction. Other studies of metaplectic tensor include [FK86, Sun97, Mez04, Tak13]. The definition of Kable has several advantages, including the variety of useful properties it enjoys and its relation to the exceptional representations (see (2.7) and (2.8) below).

For any closed subgroup $H < \mathrm{GL}_n$, Let $H^\square = \{h \in H : \det h \in F^{*2}\}$. If $\pi \in \mathrm{Alg} \widetilde{H}$, denote its restriction to \widetilde{H}^\square by π^\square .

Let α_i be a partition of n_i and $\pi_i \in \mathrm{Alg}_{\mathrm{irr}} \widetilde{M}_{\alpha_i}$ be genuine, $i = 1, 2$. Put $n = n_1 + n_2$ and let $\alpha = (\alpha_1, \alpha_2)$, i.e., α is the concatenation of α_1 and α_2 . Then

$$(\widetilde{M}_{\alpha_1}^\square, \widetilde{M}_{\alpha_2}^\square), \quad (\widetilde{M}_{\alpha_1}^\square, \widetilde{M}_{\alpha_2}), \quad (\widetilde{M}_{\alpha_1}, \widetilde{M}_{\alpha_2}^\square)$$

are pairs of commuting subgroups (see (2.1)). The representation $\pi_1^\square \otimes \pi_2^\square$ of $p^{-1}(M_{\alpha_1}^\square \times M_{\alpha_2}^\square)$ is genuine and similarly, $\pi_1^\square \otimes \pi_2$ and $\pi_1 \otimes \pi_2^\square$. Let ω be a genuine character of C_{GL_n} which agrees with

$$\omega_{\pi_1}|_{p^{-1}(C_{\mathrm{GL}_{n_1}}^2)} \otimes \omega_{\pi_2}|_{p^{-1}(C_{\mathrm{GL}_{n_2}}^2)}$$

on $p^{-1}(C_{\mathrm{GL}_n}^2)$. The metaplectic tensor $\pi_1 \widetilde{\otimes}_\omega \pi_2$ was defined by Kable [Kab01] as an irreducible summand of

$$(2.5) \quad \mathrm{ind}_{p^{-1}(M_{\alpha_1}^\square \times M_{\alpha_2}^\square)}^{\widetilde{M}_\alpha} (\pi_1^\square \otimes \pi_2^\square),$$

on which C_{GL_n} acts by ω . Note that the definition in [Kab01] was more general and included genuine admissible finite length representations, which admit a central character. Here we will only encounter the case of irreducible representations.

A more concrete description was given in [Kab01] (Corollary 3.1): if n_2 is even or n_1 and n_2 are odd, there is an irreducible summand $\sigma \subset \pi_2^\square$ such that

$$(2.6) \quad \pi_1 \widetilde{\otimes}_\omega \pi_2 = \mathrm{ind}_{p^{-1}(M_{\alpha_1} \times M_{\alpha_2}^\square)}^{\widetilde{M}_\alpha} (\pi_1 \otimes \sigma).$$

If both n_2 and n_1 are even, σ is arbitrary; if n_2 is even and n_1 is odd, σ is uniquely determined by the condition $\omega = \omega_{\pi_1} \otimes \omega_\sigma$ on C_{GL_n} ; when both are odd, $\sigma = \pi_2^\square$. Otherwise n_2 is odd and n_1 is even, the definition is similar with the roles of n_1 and n_2 reversed.

The metaplectic tensor is associative ([Kab01] Proposition 3.5) and if β_i is a subpartition of α_i , i.e., β_i is a partition of n_i with $Q_{\beta_i} < Q_{\alpha_i}$,

$$(2.7) \quad (\pi_1 \widetilde{\otimes}_\omega \pi_2)_{U_{\beta_1} \times U_{\beta_2}} = (\pi_1)_{U_{\beta_1}} \widetilde{\otimes}_\omega (\pi_2)_{U_{\beta_2}}$$

([Kab01] Proposition 4.1).

Kable did not consider the behavior of the metaplectic tensor with respect to twisted Jacquet functors, i.e., with a nontrivial character ψ . Such modules are not representations of standard Levi factors. One may attempt to extend the definition, to some extent, to

include these cases. For our purposes it will suffice to use restriction, in order to compute twisted Jacquet functors applied to the metaplectic tensor.

Claim 2.3. *Let α_i be a partition of n_i , $\pi_i \in \text{Alg}_{\text{irr}} \widetilde{M}_{\alpha_i}$ be genuine, $i = 1, 2$. Then*

$$(\pi_1 \widetilde{\otimes}_{\omega} \pi_2)|_{p^{-1}(M_{\alpha_1}^{\square} \times M_{\alpha_2})} = \begin{cases} \pi_1^{\square} \otimes \pi_2 & \text{even } n_2, \\ \pi_1^{\square} \otimes \bigoplus_{g \in \widetilde{M}_{\alpha_2}^{\square} \backslash \widetilde{M}_{\alpha_2}} \chi_g \pi_2 & \text{odd } n_1 \text{ and } n_2, \\ \bigoplus_{g \in \widetilde{M}_{\alpha_1}^{\square} \backslash \widetilde{M}_{\alpha_1}} {}^g \sigma \otimes \chi_g \pi_2 & \text{even } n_1, \text{ odd } n_2. \end{cases}$$

Here χ_g is the character of $\widetilde{M}_{\alpha_i}^{\square} \backslash \widetilde{M}_{\alpha_i}$ given by $\chi_g(x) = (\det x, \det g)_2$ and σ is an irreducible summand of π_1^{\square} . Similar results hold mutatis mutandis when restricting to $p^{-1}(M_{\alpha_1} \times M_{\alpha_2}^{\square})$. Consequently

$$(\pi_1 \widetilde{\otimes}_{\omega} \pi_2)_{p^{-1}(M_{\alpha_1}^{\square} \times M_{\alpha_2}^{\square})} = \begin{cases} [F^* : F^{*2}] \pi_1^{\square} \otimes \pi_2^{\square} & n_1 \text{ and } n_2 \text{ are odd,} \\ \pi_1^{\square} \otimes \pi_2^{\square} & \text{otherwise.} \end{cases}$$

Proof of Claim 2.3. The assertions follow from (2.6) by Mackey's theory, using [Kab01] (Propositions 3.1 and 3.2) and (2.1). For details see [Kap14b]. Note that the restriction to $p^{-1}(M_{\alpha_1}^{\square} \times M_{\alpha_2}^{\square})$ was already computed in [Kab01] (Theorem 3.1). \square

This claim has the following consequence, which will be used repeatedly.

Corollary 2.4. *Let α_i be a partition of n_i , $\pi_i, \pi'_i \in \text{Alg}_{\text{irr}} \widetilde{M}_{\alpha_i}$ be genuine for $i = 1, 2$, and $\alpha = (\alpha_1, \alpha_2)$. Assume that for an admissible representation $\tau_1 \otimes \tau_2 \in \text{Alg } M_{\alpha}$,*

$$\text{Hom}_{M_{\alpha}}((\pi_1 \widetilde{\otimes}_{\omega} \pi_2) \otimes (\pi'_1 \widetilde{\otimes}_{\omega'} \pi'_2), \tau_1 \otimes \tau_2) \neq 0.$$

Then for each i there is some square-trivial character χ_i of F^ , such that*

$$\text{Hom}_{M_{\alpha_i}}(\pi_i \otimes \pi'_i, \chi_i \tau_i) \neq 0.$$

Moreover, if n_i is even, χ_i is in fact trivial.

Proof of Corollary 2.4. According to Claim 2.3 and with the same notation, $(\pi_1 \widetilde{\otimes}_{\omega} \pi_2)|_{p^{-1}(M_{\alpha_1}^{\square} \times M_{\alpha_2})}$ is a finite direct sum of representations ${}^g \sigma \otimes \chi_g \pi_2$. Note that this form includes all the possibilities listed in the claim (e.g., σ could be π_1^{\square}), and if n_2 is even, $\chi_g = 1$. We use a similar notation ${}^{g'} \sigma' \otimes \chi_{g'} \pi'_2$ for this restriction of $\pi'_1 \widetilde{\otimes}_{\omega'} \pi'_2$. Hence for some g and g' ,

$$\text{Hom}_{M_{\alpha_1}^{\square} \times M_{\alpha_2}}(({}^g \sigma \otimes {}^{g'} \sigma') \otimes (\chi_g \pi_2 \otimes \chi_{g'} \pi'_2), \tau_1 \otimes \tau_2) \neq 0.$$

Thus if $\chi_2^{-1} = \chi_g \chi_{g'}$, $\text{Hom}_{M_{\alpha_2}}(\pi_2 \otimes \pi'_2, \chi_2 \tau_2) \neq 0$ as claimed (here we used the admissibility of τ_1 and τ_2). If n_2 is even, $\chi_g = \chi_{g'} = 1$ hence $\chi_2 = 1$. The other case is symmetric. \square

2.5. Exceptional representations. The exceptional representations were introduced and studied by Kazhdan and Patterson [KP84]. Let ξ be an exceptional character, that is, ξ is a genuine character of $C_{\widetilde{T}_n} = p^{-1}(T_n^2)C_{\widetilde{GL}_n}$, such that

$$\xi(\mathbf{s}(\text{diag}(I_{i-1}, x^2, x^{-2}, I_{n-i-1}))) = |x|, \quad \forall 1 \leq i \leq n-1, x \in F^*.$$

Let $\rho(\xi)$ be the corresponding genuine irreducible representation of \widetilde{T}_n (for $n > 1$, \widetilde{T}_n is a 2-step nilpotent subgroup). The principal series representation $\text{Ind}_{\widetilde{B}_n}^{\widetilde{GL}_n}(\delta_{B_n}^{1/2} \rho(\chi))$ has a unique irreducible quotient θ , called an exceptional representation.

The exceptional characters ξ are parameterized in the following manner. Let χ be a character of F^* . Let $\gamma : F^* \rightarrow \mathbb{C}^*$ be a mapping such that $\gamma(xy) = \gamma(x)\gamma(y)(x, y)_2^{[n/2]}$ and $\gamma(x^2) = 1$ for all $x, y \in F^*$. We call such a mapping a pseudo-character. Note that the definition of a pseudo-character depends on n , to explicate this we will occasionally call it an n -pseudo-character. Define

$$\xi_{\chi, \gamma}(\epsilon \mathfrak{s}(zI_n) \mathfrak{s}(t)) = \epsilon \gamma(z) \chi(z^n \det t) \delta_{B_n}^{1/4}(t), \quad \epsilon \in \mu_2, t \in T_n^2, z \in F^{*e}.$$

When n is even, the choice of γ is irrelevant. When $n \equiv 1 \pmod{4}$, γ is a square-trivial character of F^* . If $n \equiv 3 \pmod{4}$, $\gamma = \gamma_\psi$ for some nontrivial additive character ψ of F . Note that $\sigma(zI_n, z'I_n) = (z, z')_2^{[n/2]}$. The corresponding exceptional representation will be denoted $\theta_{n, \chi, \gamma}$, or $\theta_{\chi, \gamma}$ when n is clear from the context. Furthermore, sometimes we simply denote an exceptional representation by θ_n or θ , when the data χ and γ do not affect the validity of the argument. Note that $\omega_{\theta_{\chi, \gamma}}(\mathfrak{s}(zI_n)) = \chi(z)^n \gamma(z)$.

The following simple claim was proved in [Kap14c] (in the proof of Claim 4.1):

Claim 2.5. *We have $\theta_{\chi, \gamma} = \chi \theta_{1, \gamma}$. Additionally, if γ_0 is another pseudo-character, $\theta_{\chi, \gamma} = \eta \theta_{\chi, \gamma_0}$ for some square-trivial character η of F^* .*

Proof of Claim 2.5. The proof follows easily from the fact that γ/γ_0 is a square-trivial character of F^* . \square

The Jacquet functor carries exceptional representations into exceptional representations of Levi subgroups. The following result is due to Kable [Kab01] (Theorem 5.1),

$$(2.8) \quad (\theta_{n_1+n_2, \chi, \gamma})_{U_{n_1, n_2}} = \delta_{Q_{n_1, n_2}}^{1/4} \theta_{n_1, \chi, \gamma_1} \widetilde{\otimes}_{\chi\gamma} \theta_{n_2, \chi, \gamma_2}.$$

Here γ_1 and γ_2 are arbitrary (in [Kab01] this appears with $\chi = 1$ and with the normalized Jacquet functor). On the right-hand side $\chi\gamma$ is regarded as the character $\epsilon \mathfrak{s}(zI_n) \mapsto \epsilon \chi(z)^n \gamma(z)$.

Kazhdan and Patterson [KP84] (Section I.3, see also [BG92] p. 145 and [Kab01] Theorem 5.4) proved that for $n \geq 3$, if $|2| \neq 1$ in F , the exceptional representations do not have Whittaker models. For $n = 3$, Flicker, Kazhdan and Savin [FKS90] (Lemma 6) used global methods to extend this result to the case $|2| = 1$. The following theorem completes the proof that θ is non-generic also when $|2| = 1$ and $n > 3$.

Theorem 2.6. *For any $n \geq 3$, $\theta = \theta_n$ does not have a Whittaker model.*

Proof of Theorem 2.6. According to (2.8) (proved in [Kab01] without any assumption on the field), $\theta \subset \text{Ind}_{Q_{3, n-3}}^{\text{GL}_n} (\delta_{Q_{3, n-3}}^{1/4} \theta_3 \widetilde{\otimes} \theta_{n-3})$. Hence it is enough to prove

$$(\text{Ind}_{Q_{3, n-3}}^{\text{GL}_n} (\delta_{Q_{3, n-3}}^{1/4} \theta_3 \widetilde{\otimes} \theta_{n-3}))_{N_n, \psi} = 0.$$

By virtue of the Geometric Lemma of Bernstein and Zelevinsky ([BZ77] Theorem 5.2), this representation is glued from certain Jacquet modules of $\theta_3 \widetilde{\otimes} \theta_{n-3}$. We show that these Jacquet modules vanish, this will complete the proof. Let \mathcal{W} be a subset of Weyl elements such that $\text{GL}_n = \coprod_{w \in \mathcal{W}} Q_{3, n-3} w^{-1} N_n$. When $\psi|_{w U_{3, n-3} \cap N_n} \neq 1$, the quotient corresponding to w vanishes. This implies there is only one quotient, corresponding to $w_0 = \begin{pmatrix} & I_{n-3} \\ & \end{pmatrix}$, namely

$$(2.9) \quad \delta \cdot (\theta_3 \widetilde{\otimes} \theta_{n-3})_{N_3 \times N_{n-3}, \psi}.$$

Here δ is some modulus character and $N_3 \times N_{n-3} < M_{3,n-3}$. According to Claim 2.3, when we restrict $\theta_3 \widetilde{\otimes} \theta_{n-3}$ to $\widetilde{\text{GL}}_3^\square \times \widetilde{\text{GL}}_{n-3}^\square$, we obtain a finite direct sum of representations $\theta_3^\square \otimes \theta_{n-3}^\square$. Since the Jacquet functor with respect to N_3 and ψ commutes with this tensor and the restriction, and $(\theta_3)_{N_3, \psi} = 0$ by [FKS90] (Lemma 6), (2.9) is zero. \square

Lemma 2.7. *Assume $n \geq 3$, $\theta = \theta_n$ and let ψ be the character of $U_{1,n-1}$ given by $\psi(u) = \psi(u_{1,2})$. Then $\theta_{U_{1,n-1}, \psi}$ is a quotient of $\theta_{U_{2,n-2}}$. Similarly, $\theta_{U_{n-1,1}, \psi}$ is a quotient of $\theta_{U_{n-2,2}}$, where $\psi(u) = \psi(u_{n-1,n})$.*

Proof of Lemma 2.7. The proof closely resembles Proposition 4 of Bump, Friedberg and Ginzburg [BFG06]. We prove only the first assertion, the second is symmetric. For $1 \leq j \leq n-1$, let $V_j = U_{j,n-j} \cap M_{j-1,n-j+1}$ (e.g., $V_1 = U_{1,n-1}$ and $\prod_{j=1}^{n-1} V_j = N_n$). Let ψ be the character of $\prod_{i=1}^j V_i$ defined by $\psi(v) = \psi(\sum_{i=1}^j v_{i,i+1})$. Then $(\theta_{V_1, \psi})_{V_2, \psi} = \theta_{V_1 V_2, \psi}$ and $\text{GL}_{n-j-1} < M_{j+1,n-j-1}$ stabilizes ψ and acts transitively on the nontrivial characters of V_{j+1} . Hence it suffices to show

$$\theta_{V_1 V_2, \psi} = 0,$$

as this would imply that the action of $U_{2,n-2}$ on $\theta_{U_{1,n-1}, \psi}$ is trivial. Suppose otherwise and let $j \geq 2$ be maximal such that

$$(2.10) \quad \theta_{\prod_{i=1}^j V_i, \psi} \neq 0.$$

Since for $n \geq 3$, θ does not have a Whittaker model, we must have $j \leq n-2$. Then

$$\theta_{\prod_{i=1}^j V_i, \psi} = (\theta_{\prod_{i=1}^j V_i, \psi})_{V_{j+1}} = (\theta_{U_{j+1,n-j-1}})_{N_{j+1}, \psi} = \delta_{Q_{j+1,n-j-1}}^{1/4} (\theta_{j+1} \widetilde{\otimes} \theta_{n-j-1})_{N_{j+1}, \psi},$$

where the last equality follows from (2.8). As in the proof of Theorem 2.6 above, we use Claim 2.3 and restrict $\theta_{j+1} \widetilde{\otimes} \theta_{n-j-1}$ to $\widetilde{\text{GL}}_{j+1}^\square \times \widetilde{\text{GL}}_{n-j-1}^\square$, and since $(\theta_{j+1})_{N_{j+1}, \psi} = 0$, we obtain $(\theta_{j+1} \widetilde{\otimes} \theta_{n-j-1})_{N_{j+1}, \psi} = 0$ contradicting (2.10). \square

3. TWISTED JACQUET MODULES OF EXCEPTIONAL REPRESENTATIONS

In this section we study twisted Jacquet modules of $\theta = \theta_{n, \chi, \gamma}$. Let $U = U_{n-k, k}$, for $0 < k < n$. The Levi subgroup $M = M_{n-k, k}$ acts on the set of characters of U , with $\min(n-k, k) + 1$ orbits. We choose representatives

$$\psi_j(u) = \psi\left(\sum_{i=1}^j u_{n-k-i+1, n-k+j-i+1}\right), \quad 0 \leq j \leq \min(n-k, k).$$

For example when $n = 5$ and $k = 3$, $\psi_1(u) = \psi(u_{2,3})$ and $\psi_2(u) = \psi(u_{2,4} + u_{1,3})$. When $n = 2k$, $\psi_k \left(\begin{smallmatrix} I_k & v \\ & I_k \end{smallmatrix} \right) = \psi(\text{tr}(v))$. The stabilizer of ψ_j in M is

$$\text{St}_{n,k}(\psi_j) = \left\{ \begin{pmatrix} b & v & & \\ & c & & \\ & & c & y \\ & & & d \end{pmatrix} : b \in \text{GL}_{n-k-j}, c \in \text{GL}_j, d \in \text{GL}_{k-j} \right\}.$$

Let GL_j denote the embedding of GL_j in $\text{St}_{n,k}(\psi_j)$ via the coordinates of c . The restriction of the cover to GL_j gives a trivial double cover. In fact by (2.1) if $c, c' \in \text{GL}_j$,

$$\sigma(\text{diag}(I_{n-k-j}, c, c, I_{k-j}), \text{diag}(I_{n-k-j}, c', c', I_{k-j})) = (\det c, \det c')_2.$$

Hence any genuine representation of $\widetilde{\mathrm{GL}}_j$ takes the form $\pi \otimes \gamma_\psi$, where

$$\pi \otimes \gamma_\psi(\epsilon \mathbf{s}(\mathrm{diag}(I_{n-k-j}, c, c, I_{k-j}))) = \epsilon \gamma_\psi(\det c) \pi(c), \quad \epsilon \in \mu_2.$$

The following theorem was proved by Gelbart and Piatetski-Shapiro [GPS80] (Theorem 2.2) for the case $k = 1$. We extend it to any k .

Theorem 3.1. *Assume $n = 2k$. Then θ_{U, ψ_k} is one-dimensional and in particular, irreducible. The action of $\widetilde{\mathrm{GL}}_k$ on this space is given by the character*

$$\epsilon \mathbf{s}(\mathrm{diag}(c, c)) \mapsto \epsilon \chi^2(c) \gamma_{\psi, (-1)^{k-1}}(\det c), \quad c \in \mathrm{GL}_k.$$

Remark 3.2. *It follows that the space of functionals λ on θ such that $\lambda(\theta(u)\varphi) = \psi_k(u)\lambda(\varphi)$, where φ belongs to the space of θ and $u \in U$, is one-dimensional. In the non-metaplectic setting, if $\pi \in \mathrm{Alg} \mathrm{GL}_n$ and π_{U, ψ_k} is the trivial character, π admits a Shalika functional (see Section 2.2). Since χ^2 corresponds to $\chi^2 \otimes \gamma_\psi$ (as explained above, because the cover is trivial), it is natural to call λ a metaplectic twisted Shalika functional, or simply a metaplectic Shalika functional when $\chi = 1$. Of course it implies an embedding $\theta \subset \mathrm{Ind}_{\mathrm{GL}_k U}^{\mathrm{GL}_{2k}}(\chi^2 \gamma_{\psi, (-1)^{k-1}} \otimes \psi_k)$, which is a metaplectic twisted Shalika model. We will use this observation in Section 4. One can expect uniqueness of this model, as in the non-metaplectic setting.*

Proof of Theorem 3.1. We use induction on k . For $k = 1$ the result is known: one-dimensionality holds because then θ is generic, and the action of $\widetilde{\mathrm{GL}}_1$ is $\epsilon \mathbf{s}(\mathrm{diag}(c, c)) \mapsto \epsilon \chi^2(c) \gamma_\psi(c)$ by [GPS80] (Theorem 2.2, see also [Kab01] Lemma 5.3).

Denote the elements of U by

$$v(v_1, v_2, v_3, v_4) = \begin{pmatrix} 1 & & v_1 & v_2 \\ & I_{k-1} & v_3 & v_4 \\ & & 1 & \\ & & & I_{k-1} \end{pmatrix}.$$

By definition $\psi_k(v(v_1, v_2, v_3, v_4)) = \psi(v_1) \psi_{k-1}(v_4)$. Let

$$V_3 = \{v(0, 0, v_3, 0)\} < V, \quad V_{1,2,4} = \{v(v_1, v_2, 0, v_4)\} < V, \\ E = U_{1,k-1} = \left\{ \begin{pmatrix} 1 & e & & \\ & I_{k-1} & & \\ & & & I_k \end{pmatrix} \right\}.$$

According to the local analog of “exchanging roots”, proved by Ginzburg, Rallis and Soudry [GRS99] (Lemma 2.2, stated for unipotent subgroups of symplectic groups, but the arguments are general and hold in our setting), in $\mathrm{Alg} V_{1,2,4}$,

$$(3.1) \quad \theta_{U, \psi_k} \cong \theta_{V_{1,2,4} \rtimes E, \psi_k}.$$

Indeed, it is simple to check that the list of properties stated in the lemma holds in this setting (in the notation of [GRS99], $C = V_{1,2,4}$, $X = V_3$ and $Y = E$).

Let

$$w = \begin{pmatrix} 1 & & & \\ & \varepsilon & & \\ & & I_{k-1} & \\ & & & I_{k-1} \end{pmatrix}, \quad \varepsilon = (-1)^{k-1}.$$

Then ${}^{\mathfrak{s}(w)}(\theta_{V_{2,3,4} \rtimes E, \psi_k}) = \theta_{U', \psi'}$, where

$$U' = \left\{ \begin{pmatrix} 1 & v_1 & e & v_2 \\ & 1 & 0 & 0 \\ & & I_{k-1} & v_4 \\ & & & I_{k-1} \end{pmatrix} \right\}, \quad \psi'(u') = \psi(\varepsilon v_1) \psi_{k-1}(v_4).$$

We chose $w \in \mathrm{SL}_n$, in order to easily appeal to the formulas of Banks, Levi and Sepanski [BLS99] for computing conjugations of torus elements by $\mathfrak{s}(w)$ (see below). The actual choice of section \mathfrak{s} does not matter for the conjugation, henceforth we simply write w . Since $U' = U_{1,n-1} \rtimes U_{k-1,k-1}$, where $U_{k-1,k-1} < M_{2,2(k-1)}$, and $\psi'|_{U_{1,n-1}} = \psi(\varepsilon \cdot)$, Lemma 2.7 implies $\theta_{U', \psi'} = (\theta_{N_2 U_{2,n-2}, \psi^\circ})_{U_{k-1,k-1}, \psi_{k-1}}$, where $\psi^\circ(u) = \psi(\varepsilon u_{1,2})$ and in particular, $\psi^\circ|_{U_{2,n-2}} = 1$. This Jacquet module factors through the Jacquet module with respect to $U_{2,n-2}$ and the trivial character, hence by (2.8) it is equal to

$$\delta_{Q_{2,n-2}}^{1/4} (\theta_{2,\chi,\gamma_1} \widetilde{\otimes}_{\chi\gamma} \theta_{n-2,\chi,\gamma_2})_{N_2 \rtimes U_{k-1,k-1}, \psi(\varepsilon \cdot) \otimes \psi_{k-1}}.$$

Here γ_1 and γ_2 are arbitrary. We restrict to $\widetilde{\mathrm{GL}}_2^\square \times \widetilde{\mathrm{GL}}_{n-2}^\square$. Since $(\theta_2 \widetilde{\otimes} \theta_{n-2})^\square = \theta_2^\square \otimes \theta_{n-2}^\square$ (Claim 2.3), we obtain

$$(3.2) \quad \delta_{Q_{2,n-2}}^{1/4} ((\theta_{2,\chi,\gamma_1})_{N_2, \psi(\varepsilon \cdot)})^\square \otimes ((\theta_{n-2,\chi,\gamma_2})_{U_{k-1,k-1}, \psi_{k-1}})^\square.$$

By the induction hypothesis both spaces are one-dimensional, hence θ_{U, ψ_k} is one-dimensional.

Regarding the action of $\widetilde{\mathrm{GL}}_k$, it is of the form $\eta \otimes \gamma_\psi$ for some character η of F^* , hence it is determined by its restriction to a maximal torus.

We describe the isomorphism (3.1), in order to understand how this action is transferred from θ_{U, ψ_k} to $\theta_{V_{1,2,4} \rtimes E, \psi_k}$. The isomorphism was given in [GRS99] (proof of Lemma 2.2) using the following chain of isomorphisms:

$$\theta_{V_{1,2,4} \rtimes E, \psi_k} \rightarrow (\theta_{V_{1,2,4}, \psi_k})_E \rightarrow (\mathrm{ind}_U^{U \rtimes E} (\theta_{U, \psi_k}))_E \rightarrow \theta_{U, \psi_k}.$$

The first step was to define a mapping $\theta_{V_{1,2,4}, \psi_k} \rightarrow \mathrm{ind}_U^{U \rtimes E} (\theta_{U, \psi_k})$. For φ in the space of θ , $f(\varphi + \theta(V_{1,2,4}, \psi_k))$ in the space of $\mathrm{ind}_U^{U \rtimes E} (\theta_{U, \psi_k})$ was given by

$$f(\varphi)(x) = \theta(x)\varphi + \theta(U, \psi_k), \quad x \in U \rtimes E.$$

This mapping was extended to a mapping $\theta_{V_{1,2,4} \rtimes E, \psi_k} \rightarrow (\mathrm{ind}_U^{U \rtimes E} (\theta_{U, \psi_k}))_E$ by

$$f(\varphi + \theta(V_{1,2,4} E, \psi_k))(x) = f(\varphi)(x) + (\mathrm{ind}_U^{U \rtimes E} (\theta_{U, \psi_k}))(E).$$

To obtain an element in θ_{U, ψ_k} , integrate $f(\varphi + \theta(V_{1,2,4} E, \psi_k))(x)$ over E . Altogether

$$\varphi + \theta(V_{1,2,4} E, \psi_k) \mapsto \int_E \theta(e)\varphi \, de + \theta(U, \psi_k).$$

Let \mathbb{T}_k denote the image of T_k in GL_k . Since \mathbb{T}_k normalizes $V_{1,2,4}$, E and U , and stabilizes ψ_k , the isomorphism (3.1) extends to $\mathrm{Alg} \, \widetilde{\mathbb{T}}_k$.

Let $t = \mathrm{diag}(t_1, t_2, t_1, t_2) \in T_n$ with $t_1 \in F^*$ and $t_2 \in T_{k-1}$ (t is a general element of \mathbb{T}_k). Since

$$(3.3) \quad \theta(\mathfrak{s}(t))\varphi + \theta(V_{1,2,4} E, \psi_k) \mapsto \delta_{Q_{2,n-2}}^{1/4} ({}^w t) \theta(\mathfrak{s}(t)) \int_E \theta(e)\varphi \, de + \theta(U, \psi_k),$$

the action of $\widetilde{\mathbb{T}}_k$ on θ_{U, ψ_k} is transformed by (3.1) to ${}^{w^{-1}} \delta_{Q_{2,n-2}}^{-1/4}$ multiplied by the action of $\widetilde{\mathbb{T}}_k$ on $\theta_{V_{2,3,4} \rtimes E, \psi_k}$. The latter is given by $\theta_{V_{2,3,4} \rtimes E, \psi_k}(\mathfrak{s}(t)) = \theta_{U', \psi'}({}^w \mathfrak{s}(t))$. Using [BLS99]

(Section 2 Lemma 2, Section 3 Lemmas 3 and 1) we see that

$$\begin{aligned} {}^w \mathfrak{s}(t) &= \sigma(w, t) \mathfrak{s}({}^w t) = (t_1, \det t_2)_2 (\det t_2, \det t_2)_2 \mathfrak{s}(\text{diag}(t_1, t_1, t_2, t_2)) \\ &= (t_1, \det t_2)_2 (\det t_2, \det t_2)_2 \mathfrak{s}(\text{diag}(t_1, t_1, I_{2n-2})) \mathfrak{s}(\text{diag}(I_2, t_2, t_2)) \in \widetilde{\text{GL}}_2^\square \times \widetilde{\text{GL}}_{n-2}^\square. \end{aligned}$$

According to the induction hypothesis

$$\begin{aligned} (\theta_{2, \chi, \gamma_1})_{N_2, \psi(\varepsilon)}(\mathfrak{s}(\text{diag}(t_1, t_1))) &= \chi(t_1^2) \gamma_{\psi, \varepsilon}(t_1) = \chi(t_1^2) (t_1, t_1)_2^{k-1} \gamma_{\psi}(t_1), \\ (\theta_{n-2, \chi, \gamma_2})_{U_{k-1, k-1}, \psi_{k-1}}(\mathfrak{s}(\text{diag}(t_2, t_2))) &= \chi(\det t_2^2) \gamma_{\psi, -\varepsilon}(\det t_2) \\ &= \chi(\det t_2^2) (\det t_2, \det t_2)_2^{k-2} \gamma_{\psi}(\det t_2), \end{aligned}$$

therefore

$$\theta_{U, \psi_k}(\mathfrak{s}(t)) = \theta_{U', \psi'}({}^w \mathfrak{s}(t)) = (\chi^2 \gamma_{\psi, (-1)^{k-1}})(\det \text{diag}(t_1, t_2)).$$

Here we used (2.2) and the fact that $(-1, a)_2 = (a, a)_2$. Note that the modulus character appearing in (3.2) was cancelled by the twist of the action due to (3.3). \square

Using this result we can compute all the twisted Jacquet modules for maximal parabolic subgroups.

Theorem 3.3. *Let $0 \leq j \leq \min(n-k, k)$. We have*

$$\theta_{U, \psi_j} = \delta_{Q_{n-k-j, 2j, k-j}}^{1/4} (\theta_{n-k-j, \chi, \gamma_1} \widetilde{\otimes}_{\chi \gamma(j)} \theta_{k-j, \chi, \gamma_2}) \otimes (\theta_{2j, \chi, \gamma_3})_{U_{j, j}, \psi_j}.$$

Here the metaplectic tensor is a representation of $p^{-1}(\text{GL}_{n-k-j} \times \text{GL}_{k-j})$, where $\text{GL}_{n-k-j} \times \text{GL}_{k-j}$ is embedded in GL_n through $\text{St}_{n, k}(\psi_j)$; the pseudo-characters γ_i , $1 \leq i \leq 3$, are arbitrary; the pseudo-character $\gamma(j)$ is arbitrary when n is even, and uniquely determined by

$$(3.4) \quad \gamma(j)(z) = \gamma(z) / \gamma_{\psi, (-1)^{j-1}}(z^j), \quad \forall z \in F^*,$$

when n is odd; the right-hand side is regarded as a representation in $\text{Alg } \widetilde{\text{St}}_{n, k}(\psi_j)$ by extending it trivially on $U_{n-k-j, 2j, k-j}$.

Remark 3.4. Note that $\gamma(j)$ is really an $(n-2j)$ -pseudo-character: when j is even, $\gamma_{\psi, (-1)^{j-1}}(z^j) = 1$ and $\lfloor (n-2j)/2 \rfloor \equiv \lfloor n/2 \rfloor \pmod{2}$, while if j is odd, $\lfloor (n-2j)/2 \rfloor \equiv \lfloor (n-2)/2 \rfloor \pmod{2}$.

Proof of Theorem 3.3. For $j = 0$ this is (2.8), assume $j > 0$. We can also assume that if $n = 2k$, $j < k$, otherwise there is nothing to prove. The assertion holds for $n = 1, 2$, assume $n \geq 3$. First we claim that $U_{n-k-j, 2j, k-j}$ acts trivially on θ_{U, ψ_j} .

Claim 3.5. $\theta_{U, \psi_j} = \theta_{U_{n-k-j, j, j, k-j}, \psi_j}$.

The proof is given below. Assuming this, according to (2.8) and (2.7),

$$\begin{aligned} \theta_{U_{n-k-j, 2j, k-j}} &= \delta_{Q_{n-k-j, k+j}}^{1/4} (\theta_{n-k-j, \chi, \gamma_1} \widetilde{\otimes}_{\chi \gamma} \theta_{k+j, \chi, \gamma'})_{U_{2j, k-j}} \\ &= \delta_{Q_{n-k-j, 2j, k-j}}^{1/4} (\theta_{n-k-j, \chi, \gamma_1} \widetilde{\otimes}_{\chi \gamma} (\theta_{2j, \chi, \gamma_3} \widetilde{\otimes}_{\chi \gamma'} \theta_{k-j, \chi, \gamma_2})). \end{aligned}$$

Here $U_{2j, k-j} < \text{GL}_{k+j} < M_{n-k-j, k+j}$. Hence, noting $\psi_j|_{U_{n-k-j, 2j, k-j}} = 1$,

$$\theta_{U, \psi_j} = (\theta_{U_{n-k-j, 2j, k-j}})_{U_{j, j}, \psi_j} = \delta_{Q_{n-k-j, 2j, k-j}}^{1/4} (\theta_{n-k-j, \chi, \gamma_1} \widetilde{\otimes}_{\chi \gamma} (\theta_{2j, \chi, \gamma_3} \widetilde{\otimes}_{\chi \gamma'} \theta_{k-j, \chi, \gamma_2}))_{U_{j, j}, \psi_j},$$

where $U_{j, j} < \text{GL}_{2j} < M_{n-k-j, 2j, k-j}$.

To compute the Jacquet functor with respect to $U_{j, j}$ and ψ_j , we use the Geometric Lemma of Bernstein and Zelevinsky [BZ77] (Theorem 5.2). According to the associativity of the

metaplectic tensor ([Kab01] Proposition 3.5), and the fact that induction of a semisimple representation from a subgroup of finite index is semisimple,

$$\begin{aligned} & \theta_{n-k-j, \chi, \gamma_1} \widetilde{\Theta}_{\chi \gamma}(\theta_{2j, \chi, \gamma_3} \widetilde{\Theta}_{\chi \gamma'} \theta_{k-j, \chi, \gamma_2}) \\ &= \text{ind}_{p^{-1}(\text{GL}_{n-k-j} \times \text{GL}_{2j}^\square \times \text{GL}_{k-j})}^{\widetilde{M}_{n-k-j, 2j, k-j}} ((\theta_{n-k-j, \chi, \gamma_1} \widetilde{\Theta}_{\chi \gamma''} \theta_{k-j, \chi, \gamma_2}) \otimes \sigma), \end{aligned}$$

where $\sigma \in \text{Alg } \widetilde{\text{GL}}_{2j}^\square$ is a suitable irreducible summand of $\theta_{2j, \chi, \gamma_3}^\square$, which depends on γ'' precisely when n is odd. The double coset space $\text{GL}_{2j}^\square \backslash \text{GL}_{2j} / \text{GL}_j$ contains $[F^* : F^{*2}]$ representatives and we may take them in the form $g_z = \text{diag}(zI_j, I_j)$, where z varies over a set of representatives of $F^{*2} \backslash F^*$. These representatives normalize $\text{GL}_{n-k-j} \times \text{GL}_{2j}^\square \times \text{GL}_{k-j}$, $\text{GL}_{n-k-j} \times \text{GL}_j \times \text{GL}_{k-j}$ and $U_{j,j}$, and ${}^{g_z} \psi_j = \psi(z \cdot)_j$, i.e., ${}^{g_z} \psi_j \left(\begin{pmatrix} I_j & v \\ & I_j \end{pmatrix} \right) = \psi(z \cdot \text{tr}(v))$. Therefore

$$(\theta_{n-k-j, \chi, \gamma_1} \widetilde{\Theta}_{\chi \gamma}(\theta_{2j, \chi, \gamma_3} \widetilde{\Theta}_{\chi \gamma'} \theta_{k-j, \chi, \gamma_2}))_{U_{j,j}, \psi_j}$$

is glued from representations

$$\text{ind}_{p^{-1}(\text{GL}_{n-k-j} \times \text{GL}_j \times \text{GL}_{k-j})}^{p^{-1}(\text{GL}_{n-k-j} \times \text{GL}_{2j}^\square \times \text{GL}_{k-j})} ({}^{g_z^{-1}} ((\theta_{n-k-j, \chi, \gamma_1} \widetilde{\Theta}_{\chi \gamma''} \theta_{k-j, \chi, \gamma_2}) \otimes \sigma_{U_{j,j}, \psi(z \cdot)_j})), \quad z \in F^{*2} \backslash F^*.$$

Since $\text{GL}_j^\square = \text{GL}_j$, $\widetilde{\text{GL}}_j$ and $p^{-1}(\text{GL}_{n-k-j} \times \text{GL}_{k-j})$ commute whence each of these representations equals

$${}^{g_z^{-1}} (\theta_{n-k-j, \chi, \gamma_1} \widetilde{\Theta}_{\chi \gamma''} \theta_{k-j, \chi, \gamma_2}) \otimes {}^{g_z^{-1}} \sigma_{U_{j,j}, \psi(z \cdot)_j}.$$

The block-compatibility formula (2.1) implies (see [Kab01] p. 748)

$${}^{g_z^{-1}} (\theta_{n-k-j, \chi, \gamma_1} \widetilde{\Theta}_{\chi \gamma''} \theta_{k-j, \chi, \gamma_2}) = \chi_z \otimes (\theta_{n-k-j, \chi, \gamma_1} \widetilde{\Theta}_{\chi \gamma''} \theta_{k-j, \chi, \gamma_2}),$$

where χ_z is the non-genuine character of $\widetilde{M}_{n-k-j, k-j}$ given by $\chi_z(g) = (\det g, z^j)_2$. This is just a twist of the metaplectic tensor and will eventually be reflected on the pseudo-character changing from γ'' to $\gamma_{(j)}$. We show that ${}^{g_z^{-1}} \sigma_{U_{j,j}, \psi(z \cdot)_j}$ vanishes for all but one representative z_0 , for which it equals $(\theta_{2j, \chi, \gamma_3})_{U_{j,j}, \psi_j}$ (both representations belong in $\text{Alg } \widetilde{\text{GL}}_j$). This will complete the proof.

In general let $\pi \in \text{Alg}_{\text{irr}} \widetilde{\text{GL}}_{2j}$ be genuine. Then π^\square is the direct sum of $[F^* : F^{*2}]$ irreducible representations π_i and moreover, for each fixed summand $\rho = \pi_i$, $\pi = \text{ind}_{\widetilde{\text{GL}}_{2j}^\square}^{\widetilde{\text{GL}}_{2j}}(\rho)$ ([Kab01] Proposition 3.2). Now an application of [BZ77] (Theorem 5.2) similar to above implies that $\pi_{U_{j,j}, \psi_j}$ is glued from representations ${}^{g_z^{-1}} \rho_{U_{j,j}, \psi(z \cdot)_j}$, where z varies over the different square classes of F^* . Furthermore, since ψ and $\psi(y^2 \cdot)$ are conjugates in $\widetilde{\text{GL}}_{2j}^\square$ (use $g_{y^{-2}}$), the spaces $\rho_{U_{j,j}, \psi_j}$ and $\rho_{U_{j,j}, \psi(y^2 \cdot)_j}$ are isomorphic. Assuming $\pi_{U_{j,j}, \psi_j}$ is one-dimensional, these observations imply that for each ρ , ${}^{g_{z_0}^{-1}} \rho_{U_{j,j}, \psi(z_0 \cdot)_j} \neq 0$ for some z_0 and furthermore, z_0 is unique modulo F^{*2} with this property.

By Theorem 3.1, $(\theta_{2j, \chi, \gamma_3})_{U_{j,j}, \psi_j}$ is one-dimensional, hence for each summand σ there is a unique representative $z_0 \in F^{*2} \backslash F^*$ such that ${}^{g_{z_0}^{-1}} \sigma_{U_{j,j}, \psi(z_0 \cdot)_j} = (\theta_{2j, \chi, \gamma_3})_{U_{j,j}, \psi_j}$.

Proof of Claim 3.5. Let $V = U_{n-k-j,j}$ be embedded in $\text{St}_{n,k}(\psi_j)$ as the subgroup of matrices corresponding to the coordinates of v and denote its elements by

$$v(v_1, v_2, v_3, v_4) = \begin{pmatrix} I_{n-k-j-1} & & v_3 & v_2 \\ & 1 & v_1 & v_4 \\ & & 1 & \\ & & & I_{j-1} \end{pmatrix}.$$

The group V is abelian and normalizes U (recall that $U = U_{n-k,k}$). First we show

$$(3.5) \quad \theta_{U, \psi_j} = \theta_{UV, \psi_j}.$$

It is enough to prove $(\theta_{U, \psi_j})_{V, \mu} = 0$ for any nontrivial character μ of V . The group $\text{St}_{n,k}(\psi_j) \cap M_{n-k-j,j,j}$ (i.e., the coordinates of b and c in $\text{St}_{n,k}(\psi_j)$) acts on the characters of V and we may assume that μ does not depend on the coordinates of v_3 and v_4 , and $\mu(v(v_1, 0, 0, 0)) = \psi(v_1)$.

As in the proof of Theorem 3.1 and with a similar notation, if

$$V_3 = \{v(0, 0, v_3, 0)\} < V, \quad V_{1,2,4} = \{v(v_1, v_2, 0, v_4)\} < V, \quad E = \overline{U}_{n-k-j-1,1} < \text{GL}_{n-k-j},$$

where $\overline{U}_{n-k-j-1,1}$ is the unipotent radical opposite to $U_{n-k-j-1,1}$, Lemma 2.2 of Ginzburg, Rallis and Soudry [GRS99] implies the following isomorphism in $\text{Alg } V_{1,2,4}$,

$$(\theta_{U, \psi_j})_{V, \mu} = (\theta_{U, \psi_j})_{V_{1,2,4} \rtimes E, \mu}.$$

Conjugating the right-hand side by $\begin{pmatrix} & I_{j+1} \\ I_{n-k-j-1} & \end{pmatrix}$, we see that $(\theta_{U, \psi_j})_{V_{1,2,4} E, \mu}$ is a quotient of

$$(\theta_{U_{1,n-1}, \psi})_{U_{2,n-2} \cap U, \psi'}, \quad \psi(u) = \psi(u_{1,2}), \quad \psi'(u) = \psi(u_{2,n-k+1}),$$

where ψ was obtained from $\mu|_{\{v(v_1, 0, 0, 0)\}}$ and ψ' from ψ_j . This space vanishes according to Lemma 2.7. Hence $(\theta_{U, \psi_j})_{V, \mu} = 0$ and (3.5) is proved.

Now let $Y = U_{j,k-j}$ be embedded in $\text{St}_{n,k}(\psi_j)$ as the subgroup of matrices corresponding to the coordinates of y ,

$$y = y(y_1, y_2, y_3, y_4) = \begin{pmatrix} I_{j-1} & & y_4 & y_2 \\ & 1 & y_1 & y_3 \\ & & 1 & \\ & & & I_{k-j-1} \end{pmatrix}.$$

We show $(\theta_{UV, \psi_j})_{Y, \mu} = 0$ for any nontrivial character μ of Y . The proof is a repetition of the argument above, this time μ can be assumed to depend only on y_1 and y_2 , we take $E = \overline{U}_{1,k-j-1}$ and conjugate using $\begin{pmatrix} & I_{k-j-1} \\ I_{j+1} & \end{pmatrix}$ (embedded in the bottom right corner of GL_n). Then the result follows from Lemma 2.7 ($\theta_{U_{n-1,1}, \psi}$ is a quotient of $\theta_{U_{n-2,2}}$).

Summing up, we have shown $(\theta_{U, \psi_j}) = (\theta_{UV, \psi_j})$ and since $UVY = U_{n-k-j,j,j,k-j}$, the claim is proved. \square

\square

4. DISTINGUISHED REPRESENTATIONS

Let $\tau \in \text{Alg GL}_n$ be admissible and assume that τ admits a central character ω_τ . Given a pair of exceptional representations θ and θ' of $\widetilde{\text{GL}}_n$, we say that τ is (θ, θ') -distinguished if

$$(4.1) \quad \text{Hom}_{\text{GL}_n}(\theta \otimes \theta', \tau^\vee) \neq 0.$$

In light of Claim 2.5, the spaces $\theta_{\chi, \gamma}$ as χ and γ vary are twists of each other by characters of GL_n . Since $\theta_{\chi, \gamma} = \chi \theta_{1, \gamma}$, it is natural to fix $\chi = \chi' = 1$. Define τ to be distinguished, if it is $(\theta_{1, \gamma}, \theta_{1, \gamma'})$ -distinguished for some pair of pseudo-characters γ and γ' .

A distinguished representation τ always satisfies $\omega_\tau^2 = 1$. In particular if $\tau \in \text{Alg}_{\text{esqr}} \text{GL}_n$ is distinguished, it is already unitary whence $\tau \in \text{Alg}_{\text{esqr}} \text{GL}_n$. If $n = 1$, these are the square-trivial characters of F^* .

The following simple claim explains the motivation for removing any specific choice of pseudo-characters from the definition. Note that when n is even, the pseudo-characters are redundant because they do not affect the exceptional representations (see Section 2.5).

Claim 4.1. *If n is odd, for any square-trivial character η of F^* , τ is distinguished if and only if $\eta\tau$ is.*

Proof of Claim 4.1. For a fixed pseudo-character γ_0 , the set γ/γ_0 with γ varying over the set of pseudo-characters, exhausts all square-trivial characters of F^* . Then if τ is $(\theta_{1, \gamma}, \theta_{1, \gamma_0})$ -distinguished and $\eta = \gamma'/\gamma_0$, $\eta\theta_{1, \gamma_0} = \theta_{1, \gamma'}$ (see Claim 2.5, $\eta(z)^n = \eta(z)$ when n is odd) and $\eta\tau$ is $(\theta_{1, \gamma}, \theta_{1, \gamma'})$ -distinguished. \square

This claim is also useful for the following observation. Assume we have, for an admissible representation $\tau_1 \otimes \tau_2 \in \text{Alg } M_{n-k, k}$ (i.e., $\tau_1 \in \text{Alg GL}_{n-k}$, $\tau_2 \in \text{Alg GL}_k$ and both are admissible),

$$\text{Hom}_{M_{n-k, k}}((\theta_{n-k} \widetilde{\otimes} \theta_k) \otimes (\theta'_{n-k} \widetilde{\otimes} \theta'_k), \tau_1^\vee \otimes \tau_2^\vee) \neq 0.$$

According to Corollary 2.4, if k is even, τ_2 is distinguished. In the odd case, the corollary only implies that $\eta\tau_2$ is distinguished for some η with $\eta^2 = 1$. Then by Claim 4.1, τ_2 is distinguished also when n is odd. Similarly, we deduce that τ_1 is distinguished.

The following lemma is the application of the results of Section 3 to the study of (4.1) and will be used repeatedly below.

Lemma 4.2. *Let $\theta = \theta_{n, 1, \gamma}$, $\theta' = \theta_{n, 1, \gamma'}$ and $\tau_1 \otimes \tau_2 \in \text{Alg } M_{n-k, k}$ be an admissible representation. Assume*

$$(4.2) \quad \text{Hom}_{M_{n-k, k}}((\theta \otimes \theta')_{U_{n-k, k}}, \delta_{Q_{n-k, k}}^{1/2} \tau_1^\vee \otimes \tau_2^\vee) \neq 0.$$

Then for some $0 \leq j \leq \min(n-k, k)$,

$$\text{Hom}_{L_{n, k, j}}(\xi_{n, k, j} \otimes 1, (\delta_{Q_{n-k-j, j}}^{-1/2} (\tau_1)_{U_{n-k-j, j}} \otimes \delta_{Q_{j, k-j}}^{-1/2} (\tau_2)_{U_{j, k-j}})^\vee) \neq 0.$$

Here $L_{n, k, j} = \text{GL}_{n-k-j} \times \mathbb{GL}_j \times \text{GL}_{k-j}$, where \mathbb{GL}_j is the diagonal embedding of GL_j in $\text{St}_{n-k, k}(\psi_j)$; $\xi_{n, k, j}$ is the following representation of $p^{-1}(\text{GL}_{n-k-j} \times \text{GL}_{k-j})$,

$$\xi_{n, k, j} = (\theta_{n-k-j, 1, \gamma_1} \widetilde{\otimes}_{\gamma_{(j)}} \theta_{k-j, 1, \gamma_2}) \otimes (\theta_{n-k-j, 1, \gamma'_1} \widetilde{\otimes}_{\gamma'_{(j)}} \theta_{k-j, 1, \gamma'_2}),$$

where γ_i, γ'_i are arbitrary, $\gamma_{(j)}$ is given by (3.4) with respect to γ and the character ψ , and $\gamma'_{(j)}$ is given by (3.4) with γ' and ψ^{-1} ; 1 is the trivial character of \mathbb{GL}_j .

Proof of Lemma 4.2. By an analog of the Geometric Lemma of Bernstein and Zelevinsky ([BZ77] Theorem 5.2 and [BZ76] 5.9-5.12), in $\text{Alg } \tilde{Q}_{n-k,k}$, θ is glued from

$$\text{ind}_{\text{St}_{n,k}(\psi_j)U_{n-k,k}}^{\tilde{Q}_{n-k,k}} (\theta_{U_{n-k,k},\psi_j}), \quad 0 \leq j \leq \min(n-k, k).$$

Regarding the notation, see Section 3. According to Lemma 2.2, $(\theta \otimes \theta')_{U_{n-k,k}}$ is glued in $\text{Alg } Q_{n-k,k}$ from

$$(4.3) \quad \text{ind}_{\text{St}_{n,k}(\psi_j)U_{n-k,k}}^{Q_{n-k,k}} (\theta_{U_{n-k,k},\psi_j} \otimes \theta'_{U_{n-k,k},\psi_j^{-1}}), \quad 0 \leq j \leq \min(n-k, k).$$

Note that $U_{n-k,k}$ acts trivially on (4.3) and we regard it as a representation in $\text{Alg } M_{n-k,k}$. Also recall that by Theorem 3.3,

$$\begin{aligned} \theta_{U_{n-k,k},\psi_j} &= \delta_{Q_{n-k-j,2j,k-j}}^{1/4} (\theta_{n-k-j,1,\gamma_1} \tilde{\theta}_{\gamma(j)} \theta_{k-j,1,\gamma_2}) \otimes (\theta_{2j,1,\gamma_3})_{U_{j,j},\psi_j}, \\ \theta'_{U_{n-k,k},\psi_j^{-1}} &= \delta_{Q_{n-k-j,2j,k-j}}^{1/4} (\theta_{n-k-j,1,\gamma'_1} \tilde{\theta}'_{\gamma(j)} \theta_{k-j,1,\gamma'_2}) \otimes (\theta_{2j,1,\gamma'_3})_{U_{j,j},\psi_j^{-1}}. \end{aligned}$$

By Theorem 3.1 and because $\gamma_\psi \gamma_{\psi^{-1}} = 1$, $(\theta_{2j,1,\gamma_3})_{U_{j,j},\psi_j} \otimes (\theta_{2j,1,\gamma'_3})_{U_{j,j},\psi_j^{-1}}$ is the trivial character of \mathbb{GL}_j . Set $V = U_{n-k-j,j,k-j}$. Then $\theta_{U_{n-k,k},\psi_j} \otimes \theta'_{U_{n-k,k},\psi_j^{-1}} \in \text{Alg } (L_{n,k,j} \ltimes V)$ and is trivial on V . For $g \in L_{n,k,j}$, let $\text{mod}_V(g)$ be defined by

$$\int_V f(g^{-1}v) dv = \text{mod}_V(g) \int_V f(v) dv.$$

This is the modulus character from [BZ77] (p. 444). Then $\text{mod}_V(g) = \delta_{Q_{n-k-j,2j,k-j}}(g)$. Therefore (4.3) becomes

$$\text{ind}_{L_{n,k,j}V}^{Q_{n-k,k}} (\text{mod}_V^{1/2}(\xi_{n,k,j} \otimes 1)).$$

Assumption (4.2) implies that for some $0 \leq j \leq \min(n-k, k)$,

$$\text{Hom}_{M_{n-k,k}} (\text{ind}_{L_{n,k,j}V}^{Q_{n-k,k}} (\text{mod}_V^{1/2}(\xi_{n,k,j} \otimes 1)), \delta_{Q_{n-k,k}}^{1/2} \tau_1^\vee \otimes \tau_2^\vee) \neq 0.$$

The left-hand side equals

$$\begin{aligned} &\text{Hom}_{M_{n-k,k}} (\tau_1 \otimes \tau_2, \text{Ind}_{L_{n,k,j}V}^{Q_{n-k,k}} (\text{mod}_V^{1/2} \delta_{Q_{n-k,k}}^{-1/2} (\xi_{n,k,j} \otimes 1)^\vee)) \\ &= \text{Hom}_{L_{n,k,j}} (\text{mod}_V^{-1/2} \delta_{Q_{n-k,k}}^{1/2} (\tau_1 \otimes \tau_2)_V, (\xi_{n,k,j} \otimes 1)^\vee) \\ &= \text{Hom}_{L_{n,k,j}} (\xi_{n,k,j} \otimes 1, (\delta_{Q_{n-k-j,j}}^{-1/2} (\tau_1)_{U_{n-k-j,j}} \otimes \delta_{Q_{j,k-j}}^{-1/2} (\tau_2)_{U_{j,k-j}})^\vee). \end{aligned} \quad \square$$

Our first objective is to describe how distinguished representations are constructed. In [Kap14b] we proved the following heredity result:

Theorem 4.3. ([Kap14b] Theorem 3) *Let $\tau_1 \otimes \tau_2 \in \text{Alg } M_{n-k,k}$. If τ_1 and τ_2 are distinguished, $\tau_1 \times \tau_2$ is distinguished.*

This does not exhaust all distinguished induced representations, as can already be observed in the case of $n = 2$ and principal series representations: if τ is any character of F^* , $\tau \times \tau^{-1}$ is distinguished. The proof is contained in [Kap14b] (Claim 4.4), but was only reproduced from the arguments of Savin [Sav92]. We extend this result to the following theorem.

Theorem 4.4. *Let $\tau \in \text{Alg}_{\text{irr}} \text{GL}_k$. Then $\tau \times \tau^\vee (= \text{Ind}_{Q_{k,k}}^{\text{GL}_{2k}} (\delta_{Q_{k,k}}^{1/2} \tau \otimes \tau^\vee))$ is a distinguished representation of GL_{2k} .*

The theorem essentially follows by exhibiting a functional in

$$\mathrm{Hom}_{M_{k,k}}((\theta \otimes \theta')_{U_{k,k}}, \delta_{Q_{k,k}}^{1/2} \tau^\vee \otimes \tau).$$

We will use the metaplectic Shalika model of θ obtained in Theorem 3.1, to write an element of $(\theta \otimes \theta')_{U_{k,k}}$ as a locally constant function on GL_{2k} , bounded by a Schwartz-Bruhat function. The integral of this function against a matrix coefficient of τ will be absolutely convergent in some right half-plane, since it can be bounded using the well-known zeta integral of Godement and Jacquet [GJ72]. Then one can obtain meromorphic continuation using Bernstein's continuation principal (in [Ban98]).

Proof of Theorem 4.4. Put $Q = Q_{k,k}$, $M = M_{k,k}$, $U = U_{k,k}$ and $G = \mathrm{GL}_k$. Denote by $C_b^\infty(G)$ the space of complex-valued locally constant functions f on G that are bounded by a positive Schwartz-Bruhat function in $\mathcal{S}(F^{k \times k})$, that is, there exists $\phi \in \mathcal{S}(F^{k \times k})$ such that $|f(g)| \leq \phi(g)$ for all g . The group M acts on this space by $\mathrm{diag}(a, b)f(x) = f(b^{-1}xa)$. Let $\theta = \theta_{2k,1,\gamma}$ and $\theta' = \theta_{2k,1,\gamma'}$ (the pair γ and γ' does not matter). By Frobenius reciprocity, we need to prove

$$(4.4) \quad \mathrm{Hom}_M((\theta \otimes \theta')_U, \delta_Q^{1/2} \tau^\vee \otimes \tau) \neq 0.$$

Denote $I(\psi) = \mathrm{Ind}_{\tilde{\mathbb{G}}_U}^{\tilde{GL}_{2k}}(\gamma_{\psi,(-1)^{k-1}} \otimes \psi_k)$ and $i(\psi) = \mathrm{ind}_{\tilde{\mathbb{G}}_U}^{\tilde{Q}}(\gamma_{\psi,(-1)^{k-1}} \otimes \psi_k)$, where \mathbb{G} is the diagonal embedding of G in M . According to Theorem 3.1 (see Remark 3.2), $\theta \subset I(\psi)$. Additionally $\theta(U) \subset \theta$ in $\mathrm{Alg} \tilde{Q}$ and since $\mathrm{ind}_{\tilde{\mathbb{G}}_U}^{\tilde{Q}}(\theta_{U,\psi_k}) \subset \theta(U)$, the theorem also implies $i(\psi) \subset \theta$ (in $\mathrm{Alg} \tilde{Q}$). When we apply Theorem 3.1 to θ' using ψ^{-1} , we obtain $i(\psi^{-1}) \subset \theta' \subset I(\psi^{-1})$. First we claim:

Claim 4.5. *There exists $L \in \mathrm{Hom}_M((\theta \otimes \theta')_U, C_b^\infty(G))$, which restricts to an isomorphism $(i(\psi) \otimes i(\psi^{-1}))_U \cong \mathcal{S}(G)$ in $\mathrm{Alg} M$.*

Now let $\varphi \otimes \varphi'$ belong to the space of $\theta \otimes \theta'$. Also let h be a matrix coefficient of τ . For $s \in \mathbb{C}$, consider the integral

$$(4.5) \quad \int_G L(\varphi \otimes \varphi')(g) h(g) \nu^{s-k/2}(g) dg.$$

Since $L(\varphi \otimes \varphi') \in C_b^\infty(G)$, there is some $s_0 \in \mathbb{R}$ depending only on τ , such that (4.5) is absolutely convergent for all s with $\Re(s) > s_0$. This follows immediately from the convergence properties of the zeta integral $\int_G \phi(g) h(g) \nu^s(g) dg$ of Godement and Jacquet ([GJ72] p. 30). In this right half-plane (4.5) defines an element in

$$(4.6) \quad \mathrm{Hom}_M((\theta \otimes \theta')_U, \delta_Q^{1/2} \nu^{-s} \tau^\vee \otimes \nu^s \tau).$$

Also since $L : (i(\psi) \otimes i(\psi^{-1}))_U \rightarrow \mathcal{S}(G)$ is an isomorphism, we can choose data (φ, φ', h) such that (4.5) is absolutely convergent and equals 1, for all s . In order to deduce meromorphic continuation, we also need to show that (4.6) is at most one-dimensional, except for a finite set of values of q^{-s} , where q is the order of the residue field of F . In fact, we prove the following more general statement.

Claim 4.6. *Let $\tau_1 \otimes \tau_2 \in \mathrm{Alg}_{\mathrm{irr}} M_{n-k,k}$. Outside of a finite set of values of q^{-s} ,*

$$\mathrm{Hom}_{M_{n-k,k}}((\theta \otimes \theta')_{U_{n-k,k}}, \delta_{Q_{n-k,k}}^{1/2} \nu^{-s} \tau_1 \otimes \nu^s \tau_2)$$

is at most one-dimensional.

Now according to Bernstein's principle of meromorphic continuation and rationality (in [Ban98]), the integral (4.5) has a meromorphic continuation to a function $\Lambda(\varphi \otimes \varphi', h, s)$ in $\mathbb{C}(q^{-s})$, satisfying the same equivariance properties with respect to M , and not identically zero at $s = 0$. Assume $\Lambda(\cdot, \cdot, s)$ has a pole at $s = 0$, of order $r \geq 0$, then $\lim_{s \rightarrow 0} s^r \Lambda(\cdot, \cdot, s)$ is finite and nonzero. Now (4.4) follows and the theorem is proved.

Proof of Claim 4.5. Regard $G (= \mathrm{GL}_k)$ as a subgroup of GL_{2k} via the embedding $\ell(x) = \mathrm{diag}(x, I_k)$. In general, any $f \in I(\psi)$ defines a locally constant function on \tilde{G} by restriction. According to Lemma 3.1 of Friedberg and Jacquet [FJ93], which can easily be verified in our setting, this function is bounded by a positive Schwartz-Bruhat function in $\mathcal{S}(F^{k \times k})$ (the more general statement in [FJ93] also holds, but will not be needed here).

Define a mapping $L : I(\psi) \otimes I(\psi^{-1}) \rightarrow C_b^\infty(G)$ by

$$L(f \otimes f')(x) = f(\ell(x))f'(\ell(x)).$$

Since $\gamma_\psi \gamma_{\psi^{-1}} = 1$, we have $f(\mathrm{diag}(x, x)g)f'(\mathrm{diag}(x, x)g) = f(g)f'(g)$. Hence L intertwines the action of M and belongs to

$$\mathrm{Hom}_M(I(\psi) \otimes I(\psi^{-1}), C_b^\infty(G)).$$

Observe that L factors through $(I(\psi) \otimes I(\psi^{-1}))_U$. Indeed if $\sum_a f_a \otimes f'_a$ lies in the space of $(I(\psi) \otimes I(\psi^{-1}))(U)$, Lemma 2.1 implies in particular that for some compact $\mathcal{U} < U$,

$$\left(\sum_a f_a \otimes f'_a\right)(\ell(x)) = \int_{\mathcal{U}} \left(\sum_a f_a \otimes f'_a\right)(\ell(x)u) du = 0, \quad \forall x \in G.$$

Here we used the fact that for $x_0 \in \tilde{G}$ such that $p(x_0) = \mathrm{diag}(x, I_k) = \ell(x)$ (p - the projection $\tilde{\mathrm{GL}}_{2k} \rightarrow \mathrm{GL}_{2k}$, see Section 2.1), $f_a(x_0 \begin{pmatrix} I_k & u \\ & I_k \end{pmatrix}) = \psi_k(xu)f_a(x_0)$, and assumed $\int_{\mathcal{U}} du = 1$.

Moreover, Lemma 2.2 and its proof show that $(i(\psi) \otimes i(\psi^{-1}))_U \cong \mathrm{ind}_{\mathbb{G}U}^Q(1) \cong \mathcal{S}(G)$ and the restriction of L to $i(\psi) \otimes i(\psi^{-1}) \subset I(\psi) \otimes I(\psi^{-1})$ defines an isomorphism in

$$\mathrm{Hom}_M((i(\psi) \otimes i(\psi^{-1}))_U, \mathcal{S}(G)).$$

The claim follows because $\theta \otimes \theta' \subset I(\psi) \otimes I(\psi^{-1})$. \square

Proof of Claim 4.6. By virtue of Lemma 4.2 and using the same notation, it is enough to prove that for all $0 \leq j < \min(n-k, k)$, except for a finite set of q^{-s} ,

$$\mathrm{Hom}_{L_{n,k,j}}(\xi_{n,k,j} \otimes 1, (\nu^s \delta_{Q_{n-k-j,j}}^{-1/2} (\tau_1^\vee)_{U_{n-k-j,j}} \otimes \nu^{-s} \delta_{Q_{j,k-j}}^{-1/2} (\tau_2^\vee)_{U_{j,k-j}})^\vee) = 0,$$

and for $j = \min(n-k, k)$ this space is at most one-dimensional.

We show that this holds for each pair of irreducible subquotients $\varrho_1 \otimes \varrho_2$ of $\delta_{Q_{n-k-j,j}}^{-1/2} (\tau_1^\vee)_{U_{n-k-j,j}}$ and $\varrho_3 \otimes \varrho_4$ of $\delta_{Q_{j,k-j}}^{-1/2} (\tau_2^\vee)_{U_{j,k-j}}$. For these subquotients, this space vanishes unless both of the following spaces are nonzero:

$$(4.7) \quad \mathrm{Hom}_{\mathrm{GL}_{n-k-j} \times \mathrm{GL}_{k-j}}(\xi_{n,k,j}, \nu^{-s} \varrho_1^\vee \otimes \nu^s \varrho_4^\vee),$$

$$(4.8) \quad \mathrm{Hom}_{\mathrm{GL}_j}(\nu^s \varrho_2, \nu^s \varrho_3^\vee).$$

If $j < k$ or $j < n-k$, Corollary 2.4 (restricting (4.7) to $\mathrm{GL}_{n-k-j}^\square \times \mathrm{GL}_{k-j}$ and $\mathrm{GL}_{n-k-j} \times \mathrm{GL}_{k-j}^\square$) implies that (4.7) vanishes unless $\nu^s \varrho_1$ and $\nu^{-s} \varrho_4$ are distinguished, which is false for almost all values of q^{-s} (simply considering the central characters).

The remaining case is when $j = k = n-k$, then (4.8) becomes $\mathrm{Hom}_{\mathrm{GL}_k}(\nu^s \tau_1^\vee, \nu^s \tau_2)$, which is at most one-dimensional. \square

□

Theorems 4.3 and 4.4 imply the following corollary.

Corollary 4.7. *Let $\alpha = (\alpha_1, \dots, \alpha_m)$ be a partition of n and $\tau_1 \otimes \dots \otimes \tau_m \in \text{Alg } M_\alpha$. Assume that if α_i is odd, τ_i is distinguished, and if α_i is even, τ_i is either distinguished, or equals $\tau_{i,0} \times \tau_{i,0}^\vee$ for some $\tau_{i,0} \in \text{Alg}_{\text{irr}} \text{GL}_{\alpha_i/2}$. Then $\tau_1 \times \dots \times \tau_m$ is distinguished.*

Here and onward we assume that F is a field of characteristic 0. This is because we rely on several results not proven for fields of nonzero characteristic, e.g., certain functoriality results.

The following remark will be used in our analysis of square-integrable distinguished representations.

Remark 4.8. *Let $\tau = \langle [\nu^{(1-l)/2} \rho, \nu^{(l-1)/2} \rho] \rangle^t$, where $\rho \in \text{Alg}_{\text{cusp}} \text{GL}_k$ is unitary and $l \geq 1$ is an integer (see Section 2.2). Put $n = lk$. The computation of $L(s, \tau, \text{Sym}^2)$ by Shahidi [Sha92] (Proposition 8.1) implies the following characterization for the existence of a pole of $L(s, \tau, \text{Sym}^2)$ at $s = 0$:*

- (1) *If n is odd and ρ is self-dual, this is always the case.*
- (2) *When n is even and l is odd, $L(0, \tau, \text{Sym}^2) = \infty$ if and only if $L(0, \rho, \text{Sym}^2) = \infty$.*
- (3) *When n and l are even, $L(0, \tau, \text{Sym}^2) = \infty$ if and only if $L(0, \rho, \Lambda^2) = \infty$.*

In particular, for a pole to exist ρ (equivalently, τ) must be self-dual.

The next globalization lemma will be applied, to deduce that square-integrable representations τ such that $L(s, \tau, \text{Sym}^2)$ has a pole at $s = 0$, are distinguished. It may also be of independent interest. As mentioned in the introduction, there is an assumption here concerning the quasi-split case, see Remark 4.11 below.

Lemma 4.9. *Let $\pi \in \text{Alg}_{\text{sqf}} \text{GL}_n$. Assume that $L(s, \pi, R)$ has a pole at $s = 0$, for $R = \text{Sym}^2$ or Λ^2 . Then there exist a number field with a ring of adèles \mathbb{A} and a global cuspidal representation Π of $\text{GL}_n(\mathbb{A})$, such that for a sufficiently large finite set of places S , $L^S(s, \Pi, R)$ has a pole at $s = 1$, and at some place v , $\Pi_v = \pi$.*

Proof of Lemma 4.9. Consider the local functorial lift of [CKPSS04]. Let

$$G_n = \begin{cases} \text{SO}_{2m+1} & R = \Lambda^2, n = 2m, \\ \text{Sp}_m & R = \text{Sym}^2, n = 2m + 1, \\ \text{SO}_{2m} & R = \text{Sym}^2, n = 2m. \end{cases}$$

Here the even orthogonal group is quasi-split (split, or non-split but split over a quadratic extension). We claim that there is a generic square-integrable representation π' of G_n , such that

$$(4.9) \quad L(s, \pi' \times \varrho) = L(s, \pi \times \varrho), \quad \epsilon(s, \pi' \times \varrho, \psi) = \epsilon(s, \pi \times \varrho, \psi),$$

for any generic $\varrho \in \text{Alg}_{\text{irr}} \text{GL}_k$ and $k > 0$. Here the L and ϵ -factors are defined by the Langlands-Shahidi method [Sha90].

Indeed, if $R = \Lambda^2$ this follows immediately from Theorem 2.1 of Jiang and Soudry [JS04]. Assume $R = \text{Sym}^2$. Then if π is supercuspidal, this follows from [ACS14] and [PR12] (the appendix). In fact, the local functorial lift of an irreducible supercuspidal generic representation, of a quasi-split classical group, to some GL_N was detailed in [ACS14] (Theorem 3.2).

Given that, the results of Jiang and Soudry in the appendix of [PR12] show that the lift from quasi-split SO_{2m} is surjective onto the supercuspidal representations of GL_{2m} .

Regarding the remaining cases where $R = \text{Sym}^2$: the existence of π' when n is odd, is contained in Theorem 4.8 of Liu [Liu11]; if n is even and the central character $\omega_\pi = 1$, this was proved in Theorem 4.8 of Jantzen and Liu [JL14] (in [JL14] only the split case was analyzed, when $n = 2m$ and $\omega_\pi \neq 1$, SO_{2m} will be non-split but quasi-split). We prove the last case in the following claim.

Claim 4.10. *A representation π' with the above properties exists also when $n = 2m$, $R = \text{Sym}^2$, π is not supercuspidal and $\omega_\pi \neq 1$.*

Additionally we take $\rho \in \text{Alg}_{\text{cusp}} GL_n$ such that $L(s, \rho, R)$ has a pole at $s = 0$. By [JS04], [ACS14] and [PR12] (the appendix), there is a generic supercuspidal representation ρ' of G_n satisfying equalities similar to (4.9).

Now according to Ichino, Lapid and Mao [ILM14] (Corollary A.6, which actually holds for the types of G_n stated above, when G_n is split, by replacing Proposition A.5 with [CKPSS04] Corollary 10.1), there exist a number field with a ring of adèles \mathbb{A} and a cuspidal globally generic representation Π' of $G_n(\mathbb{A})$, such that $\Pi'_{v_1} = \pi'$ and $\Pi'_{v_2} = \rho'$ for some pair of places v_1 and v_2 .

Next we apply the global lift of [CKPSS04, CPSS11] and obtain an automorphic representation Π of $GL_n(\mathbb{A})$ such that $L^S(s, \Pi, R)$ has a pole at $s = 1$, assuming S is large enough. Moreover, the global lift preserves the γ -factors at all finite places v ([CKPSS04] Proposition 7.2), hence

$$\gamma(s, \Pi_{v_i} \times \varrho, \psi) = \gamma(s, \Pi'_{v_i} \times \varrho, \psi), \quad \forall \varrho \in \text{Alg}_{\text{cusp}} GL_k, \quad 1 \leq k \leq n-1, \quad i = 1, 2.$$

Together with the fact that the local lift preserves L and ϵ -factors, we see that

$$\gamma(s, \Pi_{v_1} \times \varrho, \psi) = \gamma(s, \pi \times \varrho, \psi), \quad \gamma(s, \Pi_{v_2} \times \varrho, \psi) = \gamma(s, \rho \times \varrho, \psi),$$

for all $\varrho \in \text{Alg}_{\text{cusp}} GL_k$ and $1 \leq k \leq n-1$. By virtue of the local converse theorem for GL_n ([Hen93] after Theorem 1.1), this implies $\Pi_{v_1} = \pi$ and $\Pi_{v_2} = \rho$. Because ρ is supercuspidal, we deduce that Π is cuspidal.

Proof of Claim 4.10. We adapt the proofs from [JS04] (Theorem 2.1) and [JL14] (Theorem 4.8). We are interested in a simpler result, concerning only square-integrable representations. Our assumptions on π are that $n = 2m$, $L(s, \pi, \text{Sym}^2)$ has a pole at $s = 0$, π is not supercuspidal and $\omega_\pi \neq 1$. Hence $\pi = \langle [\nu^{-l}\varepsilon, \nu^l\varepsilon] \rangle^t$, where $\varepsilon \in \text{Alg}_{\text{cusp}} GL_k$ is self-dual, $0 < l \in \mathbb{Z}$, k is even, $2m = (2l+1)k$ and $L(s, \varepsilon, \text{Sym}^2)$ has a pole at $s = 0$ (see Remark 4.8).

According to [ACS14] and [PR12] (the appendix), there is a generic supercuspidal representation ε' of a quasi-split SO_k satisfying

$$(4.10) \quad L(s, \varepsilon' \times \varrho) = L(s, \varepsilon \times \varrho), \quad \epsilon(s, \varepsilon' \times \varrho, \psi) = \epsilon(s, \varepsilon \times \varrho, \psi),$$

for any generic $\varrho \in \text{Alg}_{\text{irr}} GL_k$ and $k > 0$. Then $\langle [\nu\varepsilon, \nu^l\varepsilon] \rangle^t \otimes \varepsilon'$ is a representation of $GL_{lk} \times SO_k$. Consider the representation $\langle [\nu\varepsilon, \nu^l\varepsilon] \rangle^t \rtimes \varepsilon'$ of SO_{2m} , parabolically induced from $\langle [\nu\varepsilon, \nu^l\varepsilon] \rangle^t \otimes \varepsilon'$. Let π' be the unique irreducible generic subquotient of $\langle [\nu\varepsilon, \nu^l\varepsilon] \rangle^t \rtimes \varepsilon'$. One can show that π' is square-integrable (here the root system of SO_{2m} is of type B_{m-1} ; the arguments of [Mui98] Section 2 and [Tad98b], for SO_{2n+1} , can be slightly modified to apply to SO_{2m} ; in the notation of [Mui98] the pair $(\varepsilon, \varepsilon')$ satisfies (C1)). Now the proof of the preservation of local factors, i.e., (4.9), follows exactly as in [JS04] (proof of Theorem 2.1), using the multiplicativity of the γ -factors and (4.10). \square

□

Remark 4.11. *In one case, namely when $n = 2m$, $R = \text{Sym}^2$ and $\omega_\pi \neq 1$, the classical group SO_{2m} is non-split (split over a quadratic extension). In this case, the lemma is valid under two mild caveats. First, the functorial lift in the quasi-split case was only described globally ([CPSS11]), leaving out the local lift, which we use. Second, in appealing to [ILM14] (Corollary A.6), we need the following result obtained in [CKPSS04] (Corollary 10.1) for split groups: for a globally generic cuspidal representation Π' of $G_n(\mathbb{A})$, at all places v , the local Langlands parameters of Π'_v (the Satake parameters when data are unramified) are bounded in absolute value by $\frac{1}{2} - \frac{1}{N^2+1}$, where Π' functorially lifts to a representation of $\text{GL}_N(\mathbb{A})$. This result follows from the local lift and the strong bounds of Luo, Rudnick and Sarnak [LRS99], and will be obtained once the details of the local lift are provided.*

We recall Theorem 2 of [Kap14c]:

Theorem 4.12. *Let $\tau \in \text{Alg}_{\text{cusp}} \text{GL}_n$ be unitary. Then τ is distinguished if and only if $L(s, \tau, \text{Sym}^2)$ has a pole at $s = 0$.*

We extend this result to square-integrable representations. Note that in particular, a supercuspidal distinguished representation must be self-dual.

Theorem 4.13. *Let $\tau \in \text{Alg}_{\text{sqr}} \text{GL}_n$. Then τ is distinguished if and only if $L(s, \tau, \text{Sym}^2)$ has a pole at $s = 0$.*

Proof of Theorem 4.13. Write $\tau = \langle [\nu^{(1-l)/2} \rho, \nu^{(l-1)/2} \rho] \rangle^t$, where $\rho \in \text{Alg}_{\text{cusp}} \text{GL}_k$ is unitary and $l = n/k$. Assume that τ is distinguished. First we have the following claim regarding the combinatorial structure of τ .

Claim 4.14. *If τ is distinguished, then ρ is self-dual and when l is odd, ρ is distinguished.*

The claim implies that τ is self-dual, whence $L(s, \tau \times \tau)$ has a pole at $s = 0$. If n is odd, we immediately deduce $L(0, \tau, \text{Sym}^2) = \infty$ ([Sha92] Corollary 8.2, $L(0, \tau, \wedge^2) < \infty$ when n is odd). If n is even and l is odd, the claim and Theorem 4.12 imply $L(0, \rho, \text{Sym}^2) = \infty$, hence according to the second case of Remark 4.8, $L(0, \tau, \text{Sym}^2) = \infty$.

Assume n and l are even and suppose that τ is distinguished with $L(0, \tau, \text{Sym}^2) < \infty$. Then $L(0, \rho, \wedge^2) < \infty$. The claim shows that ρ is self-dual, whence $L(0, \rho, \text{Sym}^2) = \infty$.

Let G_n be the split odd general spin group of rank $n+1$ (for details see [Asg02, AS06, HS12, Kap14a]). In [Kap14a] we developed a theory of exceptional representations of G_n , parallel to the small representations of SO_{2n+1} of Bump, Friedberg and Ginzburg [BFG03, BFG06] (in the following argument one may take $G_n = \text{SO}_{2n+1}$, the only loss is in a technical requirement, for the underlying local field to contain a 4-th root of unity). These are representations of a double cover \tilde{G}_n of G_n obtained by restricting the cover of Spin_{2n+3} of Matsumoto [Mat69]. The definition of exceptional representations is similar to the definition for GL_n , and they also have a simple parametrization. If Θ and Θ' are a pair of exceptional representations of \tilde{G}_n , an admissible representation κ of G_n , which admits a central character, is called distinguished if $\text{Hom}_{G_n}(\Theta \otimes \Theta', \kappa^\vee) \neq 0$.

In [Kap14c] we proved the following “inflation” result. Let P_n be a maximal parabolic subgroup of G_n with a Levi part isomorphic to $\text{GL}_n \times \text{GL}_1$. For $\kappa_0 \in \text{Alg } \text{GL}_n$ and $s \in \mathbb{C}$, define $I(\kappa_0, s) = \text{Ind}_{P_n}^{G_n}(\delta_{P_n}^{1/2} \kappa_0 |\det|^s \otimes 1)$. According to [Kap14c] (Corollary 4.8), assuming that $\kappa_0 \in \text{Alg}_{\text{irr}} \text{GL}_n$ is tempered, $I(\kappa_0, 1/2)$ is distinguished for some pair Θ and Θ' (more

specifically described in [Kap14c], but this will not be needed here) if and only if κ_0 is distinguished.

We claim that $I(\tau, 1/2)$ is irreducible. Indeed according to Asgari, Cogdell and Shahidi [ACS14] (Proposition 4.27), because $L(0, \rho, \text{Sym}^2) = \infty$, $I(\rho, 1/2)$ is reducible and for a real $s \neq \pm 1/2$, $I(\rho, s)$ is irreducible. Then Theorem 9.1(ii) of Tadić ([Tad98a], the proof is combinatorial in nature and carries over from SO_{2n+1} to G_n) implies that $I(\tau, 1/2)$ is irreducible ($I(\tau, 1/2)$ is $\nu^{1/2}\delta(\rho, l) \rtimes 1$ in the notation of [Tad98a] and l is even).

Being irreducible, $I(\tau, 1/2)$ is also generic. Now the assumption that τ is distinguished implies that $I(\tau, 1/2)$ is a quotient of a space $\Theta \otimes \Theta'$. However, by Theorem 1 of [Kap14c], as a representation of G_n the space $\Theta \otimes \Theta'$ does not afford a Whittaker functional, contradiction.

In the opposite direction assume $L(0, \tau, \text{Sym}^2) = \infty$. By virtue of Lemma 4.9, there exist a number field k with a ring of adèles \mathbb{A} and a global cuspidal representation Π of $\text{GL}_n(\mathbb{A})$, such that $L^S(1, \Pi, \text{Sym}^2) = \infty$ and at some place v , $\Pi_v = \tau$. Bump and Ginzburg [BG92] proved that when $L^S(1, \Pi, \text{Sym}^2) = \infty$, the following period integral is nonvanishing ([BG92] Theorem 7.6)

$$(4.11) \quad \int_{Z\text{GL}_n(k)\backslash\text{GL}_n(\mathbb{A})} \varphi_\Pi(g)\phi(g)\phi'(g)dg.$$

Here Z is a subgroup of finite index in $C_{\text{GL}_n(\mathbb{A})}$ ($Z = C_{\text{GL}_n(\mathbb{A})}$ when n is odd); φ_Π is a cusp form in the space of Π ; ϕ and ϕ' are automorphic forms in the spaces of two global exceptional representations of $\text{GL}_n(\mathbb{A})$ (defined in [KP84] Section II). The local-global principle implies that τ is distinguished (see e.g. [JR92] Proposition 1).

Proof of Claim 4.14. If $l = 1$, i.e., τ is unitary supercuspidal, the result follows from Theorem 4.12. Assume $l > 1$. According to Zelevinsky [Zel80] (Proposition 9.6), $\delta_{Q_{n-k,k}}^{-1/2}(\tau^\vee)_{U_{n-k,k}} = \tau_1^\vee \otimes \nu^{(1-l)/2}\rho^\vee$ with $\tau_1 = \langle [\nu^{(1-l)/2}\rho, \nu^{(l-3)/2}\rho] \rangle^t$. Then $\tau^\vee \subset \tau_1^\vee \times \nu^{(1-l)/2}\rho^\vee$. Since τ is distinguished,

$$\text{Hom}_{\text{GL}_n}((\theta_{n,1,\gamma} \otimes \theta_{n,1,\gamma'})_{U_{n-k,k}}, \delta_{Q_{n-k,k}}^{1/2} \tau_1^\vee \otimes \nu^{(1-l)/2}\rho^\vee) \neq 0,$$

for some pair γ and γ' . Hence by Lemma 4.2, for some $0 \leq j \leq \min(n-k, k) = k$,

$$(4.12) \quad \text{Hom}_{L_{n,k,j}}(\xi_{n,k,j} \otimes 1, (\delta_{Q_{n-k-j,j}}^{-1/2}(\tau_1)_{U_{n-k-j,j}} \otimes \delta_{Q_{j,k-j}}^{-1/2}(\nu^{(l-1)/2}\rho)_{U_{j,k-j}})^\vee) \neq 0.$$

Since ρ is supercuspidal, we must have $j = 0$ or k .

If $j = 0$, the left-hand side of (4.12) becomes

$$\text{Hom}_{\text{GL}_{n-k} \times \text{GL}_k}((\theta_{n-k,1,\gamma_1} \widetilde{\otimes}_{\gamma_{(0)}} \theta_{k,1,\gamma_2}) \otimes (\theta_{n-k,1,\gamma'_1} \widetilde{\otimes}_{\gamma'_{(0)}} \theta_{k,1,\gamma'_2}), \tau_1^\vee \otimes \nu^{(1-l)/2}\rho^\vee).$$

Corollary 2.4 implies that this vanishes, unless $\nu^{(l-1)/2}\rho$ is distinguished. Since ρ is unitary, this is only possible when $l = 1$, but we are assuming $l > 1$.

Therefore we must have $j = k$. Then $\delta_{Q_{n-k-j,j}}^{-1/2}(\tau_1)_{U_{n-2k,k}} = \tau_2 \otimes \nu^{(1-l)/2}\rho$, $\tau_2 = \langle [\nu^{(3-l)/2}\rho, \nu^{(l-3)/2}\rho] \rangle^t$, and the left-hand side of (4.12) is nonzero only if both

$$\text{Hom}_{\text{GL}_{n-2k}}(\theta_{n-2k,1,\gamma_1} \otimes \theta_{n-2k,1,\gamma'_1}, \tau_2^\vee) \neq 0, \quad \text{Hom}_{\text{GL}_k}(\rho, \rho^\vee) \neq 0.$$

The second condition implies $\rho \cong \rho^\vee$. The first implies that τ_2 is distinguished, and because $\tau_2 \in \text{Alg}_{\text{sqf}} \text{GL}_{n-2k}$, we can apply induction. If l is odd, the induction terminates with a supercuspidal representation of GL_k , namely ρ , to which we apply Theorem 4.12. \square

\square

Remark 4.15. Let $\tau \in \text{Alg}_{\text{sq}} \text{GL}_n$. If $L(s, \tau, \text{Sym}^2)$ has a pole at $s = 0$, then $I(\tau, s)$ (in the notation of the proof of Theorem 4.13) is reducible at $s = 1/2$. Indeed, the theorem above implies that τ is distinguished hence by [Kap14c] (Corollary 4.8), $I(\tau, 1/2)$ is distinguished. If $I(\tau, 1/2)$ is irreducible, then it is also generic, contradicting the fact that $\Theta \otimes \Theta'$ does not afford a Whittaker functional ([Kap14c] Theorem 1). Of course this is not a new observation, as this reducibility already follows from the aforementioned result of Tadić ([Tad98a] Theorem 9.1).

We are ready to prove the characterization result for generic distinguished representations.

Theorem 4.16. Let $\tau \in \text{Alg}_{\text{irr}} \text{GL}_n$ be generic and write $\tau = \tau_1 \times \dots \times \tau_m$, where $\tau_i \in \text{Alg}_{\text{esqr}} \text{GL}_{\alpha_i}$ and $\alpha = (\alpha_1, \dots, \alpha_m)$ is a partition of n . Then τ is distinguished if and only if there is $0 \leq m_0 \leq \lfloor m/2 \rfloor$ such that, perhaps after permuting the indices of the inducing data, $\tau_{2i} = \tau_{2i-1}^\vee$ for $1 \leq i \leq m_0$ and τ_i is distinguished for $2m_0 + 1 \leq i \leq m$.

Proof of Theorem 4.16. Since τ is irreducible, one may permute the inducing data $\tau_1 \otimes \dots \otimes \tau_m$ without affecting τ . Hence one direction follows immediately from Corollary 4.7.

Assume τ is distinguished. The result is trivial for $m = 1$, assume $m > 1$. We may also assume (perhaps after applying a permutation) that $\alpha_i \leq \alpha_m$ for all $1 \leq i \leq m$. For each $1 \leq i \leq m$, set $\tau_i = \langle [\rho_i, \nu^{l_i-1} \rho_i] \rangle^t$, $l_i \geq 1$, $\rho_i \in \text{Alg}_{\text{cusp}} \text{GL}_{\alpha_i/l_i}$. Also put $k = \alpha_m$ and $l = l_m$. By Zelevinsky [Zel80] (Proposition 9.6), for any $0 \leq j \leq k$, $\delta_{Q_{j,k-j}}^{-1/2}(\tau_m)_{U_{j,k-j}} = 0$ unless k/l divides $k-j$ and then

$$\delta_{Q_{j,k-j}}^{-1/2}(\tau_m)_{U_{j,k-j}} = \langle [\nu^a \rho_m, \nu^{l-1} \rho_m] \rangle^t \otimes \langle [\rho_m, \nu^{a-1} \rho_m] \rangle^t, \quad a = l(k-j)/k.$$

Here $0 \leq a \leq l$ and since k/l divides $k-j$, $a \in \mathbb{Z}$. Note that if $a = l$ ($j = 0$), $\langle [\nu^a \rho_m, \nu^{l-1} \rho_m] \rangle^t = 1$, and when $a = 0$ ($j = k$), $\langle [\rho_m, \nu^{a-1} \rho_m] \rangle^t = 1$.

For $0 \leq j \leq n-k$, write

$$\delta_{Q_{n-k-j,j}}^{-1/2}(\tau_1 \times \dots \times \tau_{m-1})_{U_{n-k-j,j}} = \varrho_{1,j} \otimes \varrho_{2,j} \in \text{Alg } M_{n-k-j,j}.$$

Applying Lemma 4.2, for some $0 \leq j \leq \min(n-k, k)$,

$$(4.13) \quad \text{Hom}_{L_{n,k,j}}(\xi_{n,k,j} \otimes 1, (\varrho_{1,j} \otimes \varrho_{2,j} \otimes \langle [\nu^a \rho_m, \nu^{l-1} \rho_m] \rangle^t \otimes \langle [\rho_m, \nu^{a-1} \rho_m] \rangle^t)^\vee) \neq 0.$$

In particular

$$\begin{aligned} & \text{Hom}_{\text{GL}_{n-k-j} \times \text{GL}_{k-j}}((\theta_{n-k-j,1,\gamma_1} \widetilde{\otimes}_{\gamma(j)} \theta_{k-j,1,\gamma_2}) \otimes (\theta_{n-k-j,1,\gamma'_1} \widetilde{\otimes}_{\gamma'(j)} \theta_{k-j,1,\gamma'_2}), \\ & \quad \varrho_{1,j}^\vee \otimes (\langle [\rho_m, \nu^{a-1} \rho_m] \rangle^t)^\vee) \neq 0 \end{aligned}$$

and according to Corollary 2.4, $\varrho_{1,j}$ and $\langle [\rho_m, \nu^{a-1} \rho_m] \rangle^t$ are distinguished.

If $j = 0$, $\varrho_{1,j} = \tau_1 \times \dots \times \tau_{m-1}$ is a distinguished generic representation and $\langle [\rho_m, \nu^{a-1} \rho_m] \rangle^t = \tau_m$. The proof is then complete, by applying induction to $\tau_1 \times \dots \times \tau_{m-1}$.

Henceforth assume $j > 0$ and then $a < l$. If also $j < k$, then since $\langle [\rho_m, \nu^{a-1} \rho_m] \rangle^t \in \text{Alg}_{\text{esqr}} \text{GL}_{k-j}$ is distinguished, Theorem 4.13 implies $\langle [\rho_m, \nu^{a-1} \rho_m] \rangle^t = \langle [\nu^{(1-a)/2} \rho, \nu^{(a-1)/2} \rho] \rangle^t$ for a self-dual $\rho \in \text{Alg}_{\text{cusp}} \text{GL}_{k/l}$. Put $A = (a-1)/2$. Then $\rho_m = \nu^{-A} \rho$,

$$\begin{aligned} \tau_m &= \langle [\nu^{-A} \rho, \nu^{-A+l-1} \rho] \rangle^t, \\ \delta_{Q_{j,k-j}}^{-1/2}(\tau_m)_{U_{j,k-j}} &= \langle [\nu^{A+1} \rho, \nu^{-A+l-1} \rho] \rangle^t \otimes \langle [\nu^{-A} \rho, \nu^A \rho] \rangle^t \end{aligned}$$

and note that $A+1 \leq -A+l-1$ and $A \geq 0$ (because $0 < a < l$). When $j = k$, we cannot deduce anything about the structure of τ_m .

To proceed we need to describe the structure of $\varrho_{1,j} \otimes \varrho_{2,j}$. To this end we use the concrete description of the composition factors of $\delta_{Q_{n-k-j,j}}^{-1/2} (\tau_1 \times \dots \times \tau_{m-1})_{U_{n-k-j,j}}$ given by Zelevinsky ([Zel80] 1.6), which is a formulation of [BZ77] (Theorem 5.2) for this setting. Let \mathcal{B} be the set of $(m-1) \times 2$ matrices whose coordinates are nonnegative integers, the sum of the coordinates in the i -th row is α_i , the sum of the coordinates in the first column is $n-k-j$, and the second column sums up to j . Let $b \in \mathcal{B}$. The (i, r) -th coordinate of b will be denoted $b(i, r)$. For each $1 \leq i \leq m-1$, $(\tau_i)_{U_{b(i,1),b(i,2)}} = 0$ unless α_i/l_i divides $b(i, 2)$ and then

$$\delta_{Q_{b(i,1),b(i,2)}}^{-1/2} (\tau_i)_{U_{b(i,1),b(i,2)}} = \langle [\nu^{a_i} \rho_i, \nu^{l_i-1} \rho_i] \rangle^t \otimes \langle [\rho_i, \nu^{a_i-1} \rho_i] \rangle^t, \quad a_i = l_i b(i, 2)/\alpha_i.$$

Note that if $b(i, 2) = \alpha_i$, $\langle [\nu^{a_i} \rho_i, \nu^{l_i-1} \rho_i] \rangle^t = 1$ and when $b(i, 2) = 0$, $\langle [\rho_i, \nu^{a_i-1} \rho_i] \rangle^t = 1$. Assuming α_i/l_i divides $b(i, 2)$ for each $1 \leq i \leq m-1$, put

$$\begin{aligned} \varrho_{1,j,b} &= \langle [\nu^{a_1} \rho_1, \nu^{l_1-1} \rho_1] \rangle^t \times \dots \times \langle [\nu^{a_{m-1}} \rho_{m-1}, \nu^{l_{m-1}-1} \rho_{m-1}] \rangle^t, \\ \varrho_{2,j,b} &= \langle [\rho_1, \nu^{a_1-1} \rho_1] \rangle^t \times \dots \times \langle [\rho_{m-1}, \nu^{a_{m-1}-1} \rho_{m-1}] \rangle^t. \end{aligned}$$

Then $\delta_{Q_{n-k-j,j}}^{-1/2} (\tau_1 \times \dots \times \tau_{m-1})_{U_{n-k-j,j}}$ is glued from the representations

$$\varrho_{1,j,b} \otimes \varrho_{2,j,b}, \quad b \in \mathcal{B}.$$

The following claim uses this description to analyze the structure of $\varrho_{1,j,b}$ and $\varrho_{2,j,b}$, given that $\varrho_{2,j,b}$ contains a certain quotient.

Claim 4.17. *Let $\pi \in \text{Alg}_{\text{cusp}} \text{GL}_{k/l}$ and assume that $0 < j \leq k$, $lj/k \in \mathbb{Z}$ and for some $b \in \mathcal{B}$,*

$$(4.14) \quad \text{Hom}_{\text{GL}_j}(\varrho_{2,j,b}, \langle [\pi, \nu^{\frac{lj}{k}-1} \pi] \rangle^t) \neq 0.$$

If $j < k$, then $\nu^{a_i-1} \rho_i = \nu^{\frac{lj}{k}-1} \pi$ for some i with $b(i, 2) > 0$. If $j = k$, then for some $1 \leq i \leq m-1$, $\tau_i \cong \langle [\pi, \nu^{l-1} \pi] \rangle^t$, and

$$\varrho_{1,k,b} = \tau_1 \times \dots \times \tau_{i-1} \times \tau_{i+1} \times \dots \times \tau_{m-1}.$$

Now we can dispose of the nontrivial non-generic orbits. Indeed, assume $j < k$. Suppose (4.13) does not vanish, then for some $b \in \mathcal{B}$,

$$\text{Hom}_{\text{GL}_j}(\varrho_{2,j,b}, \langle [\nu^{A-l+1} \rho, \nu^{-A-1} \rho] \rangle^t) \neq 0.$$

Applying Claim 4.17 we deduce that $\nu^{a_i-1} \rho_i = \nu^{-A-1} \rho$ for some i with $b(i, 2) > 0$. Then

$$\tau_i = \langle [\nu^{-A-a_i} \rho, \nu^{-A-a_i+l_i-1} \rho] \rangle^t, \quad \tau_m = \langle [\nu^{-A} \rho, \nu^{-A+l-1} \rho] \rangle^t.$$

We show that τ_i and τ_m must be linked, contradicting the irreducibility of τ . Specifically, we show

$$-A - a_i < -A, \quad -A - a_i + l_i - 1 < -A + l - 1, \quad -A - a_i + l_i - 1 \geq -A - 1.$$

Because $\rho_i \in \text{Alg GL}_{\alpha_i/l_i}$ and $\rho \in \text{Alg GL}_{k/l}$, we have $\alpha_i/l_i = k/l$, and since (by assumption) $\alpha_i \leq k$, we obtain $l_i \leq l$. The first two inequalities follow from this and because $a_i = l_i b(i, 2)/\alpha_i > 0$. The last holds because $b(i, 2) \leq \alpha_i$ (by definition) whence $a_i \leq l_i$.

Finally, assume $j = k$. Then (4.13) vanishes unless for some $b \in \mathcal{B}$, both

$$\text{Hom}_{\text{GL}_{n-2k}}(\theta_{n-2k,1,\gamma_1} \otimes \theta_{n-2k,1,\gamma'_1}, \varrho_{1,k,b}^\vee) \neq 0, \quad \text{Hom}_{\text{GL}_k}(\varrho_{2,k,b}, \tau_m^\vee) \neq 0.$$

Again appealing to Claim 4.17 we find that $\tau_i \cong \tau_m^\vee$ for some $1 \leq i \leq m-1$ and

$$\varrho_{1,k,b} = \tau_1 \times \dots \times \tau_{i-1} \times \tau_{i+1} \times \dots \times \tau_{m-1}.$$

If $n = 2k$, we are done. Otherwise the theorem follows by applying induction to $\varrho_{1,k,b}$.

Proof of Claim 4.17. Let $1 \leq i_1 < \dots < i_c \leq m-1$ be a minimal set of indices such that $b(i_1, 2) + \dots + b(i_c, 2) = j$, $c \geq 1$. Then

$$\varrho_{2,j,b} = \langle [\rho_{i_1}, \nu^{a_{i_1}-1} \rho_{i_1}] \rangle^t \times \dots \times \langle [\rho_{i_c}, \nu^{a_{i_c}-1} \rho_{i_c}] \rangle^t.$$

In general if $\varepsilon \in \text{Alg GL}_n$ is a subrepresentation of a representation parabolically induced from a cuspidal representation $\varepsilon_1 \otimes \dots \otimes \varepsilon_t \in \text{Alg}_{\text{cusp}} M_\beta$, where β is a partition of n into t parts, we say that the cuspidal support of ε is the multiset (i.e., including multiplicities) $\{\varepsilon_1, \dots, \varepsilon_t\}$ and denote it by $\text{Supp}(\varepsilon)$. According to Zelevinsky [Zel80] (9.1 and 6.1)

$$\text{Supp}(\varrho_{2,j,b}) = \{\rho_{i_r}, \dots, \nu^{a_{i_r}-1} \rho_{i_r}\}_{1 \leq r \leq c}, \quad \text{Supp}(\langle [\pi, \nu^{\frac{lj}{k}-1} \pi] \rangle^t) = \{\pi, \dots, \nu^{\frac{lj}{k}-1} \pi\}.$$

Assumption (4.14) implies that $\langle [\pi, \nu^{\frac{lj}{k}-1} \pi] \rangle^t$ is an irreducible subquotient of a representation induced from the tensor of all the representations (with multiplicities) appearing in $\text{Supp}(\varrho_{2,j,b})$. Hence by Bernstein and Zelevinsky [BZ77] (Theorem 2.9), as multisets

$$(4.15) \quad \text{Supp}(\varrho_{2,j,b}) = \{\pi, \dots, \nu^{\frac{lj}{k}-1} \pi\}.$$

Therefore $\rho_{i_r} = \nu^{s_{i_r}} \pi$ for some integer $s_{i_r} \geq 0$, for all r . Furthermore, there exists $i \in \{i_1, \dots, i_c\}$ such that $\nu^{a_i-1} \rho_i = \nu^{\frac{lj}{k}-1} \pi$ and clearly $b(i, 2) > 0$. This completes the proof for $j < k$.

Henceforth assume $j = k$. If $c = 1$, then for $i = i_1$ ($1 \leq i \leq m-1$), $\alpha_i \geq b(i, 2) = k$. Since $k \geq \alpha_i$, we see that $b(i, 2) = \alpha_i$, $\varrho_{2,k,b} = \tau_i$ and (4.14) implies $\tau_i \cong \langle [\pi, \nu^{l-1} \pi] \rangle^t$. Also because $b(i, 1) + b(i, 2) = \alpha_i$ and the sum of entries in the second column of b is k , we have $b(i, 1) = 0$, $b(r, 2) = 0$ and $b(r, 1) = \alpha_r$ for all $r \neq i$. Thus

$$\varrho_{1,k,b} = \tau_1 \times \dots \times \tau_{i-1} \times \tau_{i+1} \times \dots \times \tau_{m-1},$$

as claimed.

Assume $c > 1$. For simplicity, assume $i_r = r$ for $1 \leq r \leq c$. Now we can denote, for $1 \leq i \leq c$,

$$[\rho_i, \nu^{l_i-1} \rho_i] = [s_i, e_i], \quad [\rho_i, \nu^{a_i-1} \rho_i] = [s_i, h_i], \quad s_i \leq h_i \leq e_i, \quad s_i, h_i, e_i \in \mathbb{Z}.$$

Indeed, the representations ρ_i are all unramified twists of one representation π , so ρ_i can be safely excluded from the notation. In other words, the segment $\{\rho_i, \dots, \nu^{l_i-1} \rho_i\}$ will henceforth be denoted $[s_i, e_i]$. Note that $\tau_i = \langle [\rho_i, \nu^{l_i-1} \rho_i] \rangle^t = \langle [\nu^{s_i} \pi, \nu^{e_i} \pi] \rangle^t$. We can rewrite (4.15) in the form

$$(4.16) \quad \{\nu^{s_i} \pi, \dots, \nu^{h_i} \pi\}_{1 \leq i \leq c} = \{\pi, \dots, \nu^{l-1} \pi\}.$$

In particular $0 \leq s_i \leq l-1$ for all i . Renumbering if necessary, we can assume that the segment $[s_1, e_1]$ is such that $s_1 = 0$ and $e_i \leq e_1$ for all i such that $s_i = 0$.

Because $\rho_i \in \text{Alg GL}_{\alpha_i/l_i}$ and $\pi \in \text{Alg GL}_{k/l}$, we have $\alpha_i/l_i = k/l$, and since (by assumption) $\alpha_i \leq k$, we obtain $l_i \leq l$. Since $e_i - s_i + 1 = l_i \leq l$ and $s_1 = 0$, $e_1 \leq l-1$. If $s_1 < s_i \leq e_1 + 1$, then $s_i \leq e_i \leq e_1$ because otherwise the segments $[s_1, e_1]$ and $[s_i, e_i]$ are linked contradicting the fact that τ is irreducible. In particular, $s_i \neq e_1 + 1$ for all i . If $e_1 < l-1$, $\nu^{e_1+1} \pi$ must appear in $\text{Supp}(\varrho_{2,j,b})$, but this is impossible because $e_1 + 1$ is not covered by any segment $[s_i, e_i]$. Therefore $e_1 = l-1$, then $s_i \leq e_1$, whence $e_i \leq e_1$ for all i .

Thus far we have shown $[s_1, e_1] = [0, l-1]$, which immediately implies $\tau_1 = \langle [\pi, \nu^{l-1} \pi] \rangle^t$, and all other segments are contained in the first segment. Next we argue that the segments

$[s_i, h_i]$ are disjoint. Indeed, any such segment contributes at most $h_i - s_i + 1 = a_i = l_i b(i, 2)/\alpha_i$ representations to $\text{Supp}(\varrho_{2,j,b})$ and since $l_i/\alpha_i = l/k$,

$$\sum_{i=1}^c h_i - s_i + 1 = \frac{l}{k} \sum_{i=1}^c b(i, 2) = l.$$

But $\text{Supp}(\langle [\pi, \nu^{l-1}\pi] \rangle^t)$ contains l non-isomorphic representations, hence by (4.16) the segments are disjoint.

Therefore, perhaps after renumbering segments $2, \dots, c$, we can assume $s_1 < s_2 < \dots < s_c$. Then the fact that their lengths sum up to l also implies $s_{i+1} = h_i + 1$ whence

$$s_1 \leq h_1 < s_2 \leq h_2 < s_3 \leq h_3 < \dots < s_c \leq h_c.$$

Because the segments $[s_i, e_i]$, $[s_{i+1}, e_{i+1}]$ are not linked and $s_i < s_{i+1} \leq e_i + 1$ ($h_i \leq e_i$), we obtain $e_c \leq \dots \leq e_2 \leq e_1 \leq l - 1$. Now we must have $e_c = l - 1$, because $\nu^{l-1}\pi \in \text{Supp}(\varrho_{2,j,b})$ only if $h_c \geq l - 1$. It follows that $h_c = e_c$ and $e_1 = \dots = e_c$.

We conclude $[h_i + 1, e_i] = [s_{i+1}, e_{i+1}]$ for all $1 \leq i \leq c - 1$ and

$$\langle [\nu^{h_1+1}\pi, \nu^{e_1}\pi] \rangle^t \times \dots \times \langle [\nu^{h_{c-1}+1}\pi, \nu^{e_{c-1}}\pi] \rangle^t = \langle [\nu^{s_2}\pi, \nu^{e_2}\pi] \rangle^t \times \dots \times \langle [\nu^{s_c}\pi, \nu^{e_c}\pi] \rangle^t = \tau_2 \times \dots \times \tau_c.$$

For the last segment $[s_c, h_c] = [s_c, e_c]$, so $b(c, 1) = 0$ and

$$\varrho_{1,j,b} = \langle [\nu^{h_1+1}\pi, \nu^{e_1}\pi] \rangle^t \times \dots \times \langle [\nu^{h_{c-1}+1}\pi, \nu^{e_{c-1}}\pi] \rangle^t \times \tau_{c+1} \times \dots \times \tau_{m-1} = \tau_2 \times \dots \times \tau_{m-1}.$$

Together with $\tau_1 = \langle [\pi, \nu^{l-1}\pi] \rangle^t$ (proved above), the proof is complete also when $j = k$ and $c > 1$. Note that the indices were permuted during the argument, in the original representation τ_1 will actually be τ_i for some $1 \leq i \leq m - 1$. \square

\square

Remark 4.18. *The only place in the proof where we assumed that the characteristic of the field is 0, was in the application of Theorem 4.13. However, it is simple to see that we only used the fact that a square-integrable distinguished representation is self-dual. Granted this, Theorem 4.16 is valid also for fields of odd characteristic.*

Corollary 4.19. *Let $\tau \in \text{Alg}_{\text{irr}} \text{GL}_n$ be a generic distinguished representation. Then τ is self-dual.*

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